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FUZZY POWER AGGREGATION OPERATORS AND THEIR APPLICATION TO MULTIPLE ATTRIBUTE GROUP DECISION MAKING

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Abstract. The article investigates the multiple attribute group decision making (MAGDM) problems in which the attribute values take the form of triangular fuzzy information. Motivated by the ideal of power aggregation, in this paper some power aggregation operators for aggregating triangular fuzzy information are developed and then applied in order to develop some models for multiple attribute group decision making with triangular fuzzy information. Finally, some illustrative examples are given to verify the developed approach and to demonstrate its practicality and effectiveness.

Keywords: multiple attribute group decision making (MAGDM), triangular fuzzy numbers, power average operator, power geometric operator, fuzzy power aggregation operator.

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Introduction

A multiple attribute decision making problem is to find a desirable solution from a finite number of feasible alternatives assessed on multiple attributes, both quantitative and qualitative (Liu 2009; Zhang, Liu 2010, 2010b; Wei 2012; Xu, Cai 2012; Xu 2011; Xu, Wang 2011). Group decision-making (i.e. multi-expert) is a typical decision-making activity, where utilizing several experts alleviate some of the decision-making difficulties due to the complexity and uncertainty of the problem. Group decision-making problems usually follow a common

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resolution scheme composed by two phases: aggregation phase and exploitation phase (Herrera, Herrera-Viedma 2000). However, under many conditions, for the real multiple attribute decision making problems, the decision information about alternatives is usually uncertain or fuzzy due to the increasing complexity of the socio-economic environment and the vagueness of inherent subjective nature of human thought; thus, numerical values are inadequate or insufficient to model real-life decision problems.

In the literature, many aggregation operators and approaches have been developed to solve the multiple attribute group decision-making problems with fuzzy information. Xu (2003), Wang and Fan (2003) developed the fuzzy ordered weighted averaging (FOWA) operator. Xu (2002) introduced the fuzzy ordered weighted geometric (FOWG) operator. Xu and Wu (2004) proposed the fuzzy induced ordered weighted averaging (FIOWA) operator. Xu and Da (2003) developed the fuzzy induced ordered weighted geometric (FIOWG) operator. Xu (2009) developed some fuzzy harmonic mean operators, such as fuzzy weighted harmonic mean (FWHM) operator, fuzzy ordered weighted harmonic mean (FOWHM) operator, fuzzy hybrid harmonic mean (FHHM) operator. Wei (2009c) proposed fuzzy ordered weighted harmonic mean (FOWHM) operator. Wei (2011a) developed the fuzzy induced ordered weighted harmonic mean (FIOWHM) operator and applied FIOWHM operator to multiple attribute group decision making.

However, all these aggregation operators and approaches do not take into account information about the relationship between the triangular fuzzy variables being aggregated. To overcome this drawback, motivated by the ideal of power aggregation (Yager 2001), in this paper some fuzzy power aggregation operators are proposed: the fuzzy power weighted average (FPWA) operator, the fuzzy power weighted geometric (FPWG) operator, fuzzy power weighted harmonic average (FPWHA), fuzzy power weighted quadratic average (FPWQA), the fuzzy power ordered weighted average (FPOWA) operator, the fuzzy power ordered weighted geometric (FPOWG) operator, fuzzy power ordered weighted harmonic average (FPOWHA) and fuzzy power ordered weighted quadratic average (FPOWQA). The prominent characteristic of these operators is that they take into account information about the relationship between the triangular fuzzy variables being aggregated. Then, based on these triangular fuzzy power aggregation operators, some approaches to multiple attribute group decision making problems with triangular fuzzy information were developed which can avoid the subjectivity of the decision makers' information weights.

1. Preliminaries

1.1. Triangular fuzzy numbers

In the following, there are briefly described some basic concepts and basic operational laws related to the triangular fuzzy numbers.

Definition 1 (Van Laarhoven, Pedrycz 1983)**.** A triangular fuzzy numbers $\tilde{a}$ can be defined by a triplet $\left(a^L, a^M, a^U\right)$. The membership function $\mu_{\tilde{a}}\left(x\right)$ is defined as:

$$\mu_{\tilde{a}}(x)=\begin{cases}0, & x<a^L,\\ \dfrac{x-a^L}{a^M-a^L}, & a^L\le x\le a^M,\\ \dfrac{x-a^U}{a^M-a^U}, & a^M\le x\le a^U,\\ 0, & x\ge a^U,\end{cases} \tag{1}$$

where $0<a^L\le a^M\le a^U$, a^L and a^U stand for the lower and upper values of the support of $\tilde{a}$, respectively, and a^M for the modal value.

Definition 2 (Van Laarhoven, Pedrycz 1983**).** Basic operational laws related to triangular fuzzy numbers:

$$\tilde{a}\oplus\tilde{b}=\left[a^L,a^M,a^U\right]\oplus\left[b^L,b^M,b^U\right]=\left[a^L+b^L,a^M+b^M,a^U+b^U\right],$$

$$\tilde{a}\otimes\tilde{b}=\left[a^L,a^M,a^U\right]\otimes\left[b^L,b^M,b^U\right]=\left[a^Lb^L,a^Mb^M,a^Ub^U\right],$$

$$\lambda\otimes\tilde{a}=\lambda\otimes\left[a^L,a^M,a^U\right]=\left[\lambda a^L,\lambda a^M,\lambda a^U\right],\ \lambda>0.$$

$$\frac{1}{\tilde{a}}=\left[1/a^U,1/a^M,1/a^L\right].$$

Zétényi (1998) pointed out that psychologists generally consider a good representation of a fuzzy set its expected value. The expected value of a fuzzy set A is equal to (Matarazzo, Munda 2001):

$$E(A)=\frac{\int_{-\infty}^{+\infty}x\mu_A(x)dx}{\int_{-\infty}^{+\infty}\mu_A(x)dx}, \tag{2}$$

where the integral converges absolutely, that is $\int_{-\infty}^{+\infty}\left|x\mu_A(x)\right|dx<+\infty$. Otherwise, A has no finite expected value.

Definition 3. If A is a triangular fuzzy variable $\left[a^L,a^M,a^U\right]$, according to the Eqn (2), then the expected value of A is:

$$E(A)=\frac{1}{3}\left(a^L+a^M+a^U\right). \tag{3}$$

1.2. Power aggregation operator

Yager (2001) developed a nonlinear weighted average aggregation operator called power average (PA) operator, which can be defined as follows:

$$PA(a_1,a_2,\cdots,a_n)=\frac{\sum_{i=1}^{n}\left(1+T(a_i)\right)a_i}{\sum_{i=1}^{n}\left(1+T(a_i)\right)}, \tag{4}$$

where $T(a_i)=\sum_{\substack{j=1\\ j\ne i}}^{n}Sup(a_i,a_j)$, and $Sup(a,b)$ is the support for a from b, which satisfies the following three properties: (1) $Sup(a,b)\in[0,1]$; (2) $Sup(a,b)=Sup(b,a)$; (3) $Sup(a,b)\geq Sup(x,y)$, if $|a-b|<|x-y|$. Obviously, the support (Sup) measure is essentially a similarity index, that is, the more similar, the closer two values, and the more they support each other.

Based on the PA operator and geometric mean, in the following, Xu and Yager (2010) further define a power geometric (PG) operator:

$$PG(a_1,a_2,\cdots,a_n)=\prod_{i=1}^{n} a_i^{\frac{1+T(a_i)}{\sum_{i=1}^{n}(1+T(a_i))}}. \tag{5}$$

Obviously, the PA and PG operators are two nonlinear weighted aggregation tools, whose weighting vectors depend upon the input values and allow values being aggregated to support and reinforce each other, that is to say, the closer a_i and a_j, the more similar they are, and the more they support each other.

2. Fuzzy power aggregation operators

2.1. FPWA operator and FPWG operator

The PA (Yager 2001) and PG (Xu, Yager 2010) operators, however, have usually been used in situations where the input arguments are the exact values. Here the PA and PG operators should be extended to accommodate the situations where the input arguments are triangular fuzzy information. In the following, some fuzzy power aggregation operators should be developed, which allows the input data to support each other in the aggregating process.

Definition 4. Let $\tilde{a}_i=\left[a_i^L,\ a_i^M,\ a_i^U\right](i=1,2,\cdots,n)$ be a set of triangular fuzzy numbers and $\omega=(\omega_1,\omega_2,\cdots,\omega_n)^T$ be the weighting vector of $\tilde{a}_i\ (i=1,2,\cdots,n)$ and $\omega_i\in[0,1]$, $\sum_{i=1}^{n}\omega_i=1$, then we define the fuzzy power weighted average (FPWA)operator as follows:

$$FPWA_{\omega}(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)=\frac{\sum_{i=1}^{n}\omega_i\left(1+T(\tilde{a}_i)\right)\tilde{a}_i}{\sum_{i=1}^{n}\omega_i\left(1+T(\tilde{a}_i)\right)}, \tag{6}$$

where:

$$T(\tilde{a}_i)=\sum_{\substack{j=1\\ j\neq i}}^{n}\omega_j Sup(\tilde{a}_i,\tilde{a}_j) \tag{7}$$

and $Sup(\tilde{a}_i,\tilde{a}_j)$ is the support for $\tilde{a}_i$ from $\tilde{a}_j$, with the conditions:

1) $Sup(\tilde{a}_i,\tilde{a}_j)\in[0,1]$;

2) $Sup(\tilde{a}_i,\tilde{a}_j)=Sup(\tilde{a}_j,\tilde{a}_i)$;

3) $Sup(\tilde{a}_i,\tilde{a}_j)\geq Sup(\tilde{a}_s,\tilde{a}_t)$, if $d(\tilde{a}_i,\tilde{a}_j)\leq d(\tilde{a}_s,\tilde{a}_t)$, where d is a distance measure.

Especially, if $\omega = \left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right)^T$, then the FPWA operator reduces to a fuzzy power average (FPA) operator:

$$FPA_{\omega}\left(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n\right) = \frac{\sum_{i=1}^{n}\left(1 + T\left(\tilde{a}_i\right)\right)\tilde{a}_i}{\sum_{i=1}^{n}\left(1 + T\left(\tilde{a}_i\right)\right)}, \tag{8}$$

where:

$$T\left(\tilde{a}_i\right) = \frac{1}{n}\sum_{\substack{j=1 \\ j \neq i}}^{n} Sup\left(\tilde{a}_i, \tilde{a}_j\right). \tag{9}$$

It can be easily proved that the FPWA operator has the following properties.

Theorem 1. (Idempotency) If $\left(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n\right) = \tilde{a}$, then:

$$FPWA_{\omega}\left(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n\right) = \tilde{a}. \tag{10}$$

Theorem 2. (Boundedness).

$$\min_i \tilde{a}_i \leq FPWA_{\omega}\left(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n\right) \leq \max_i \tilde{a}_i.$$

Based on the FPWA operator and the geometric mean, here we define a fuzzy power weighted geometric (FPWG) operator:

$$FPWG_{\omega}\left(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n\right) = \prod_{i=1}^{n}\left(\tilde{a}_i\right)^{\frac{\omega_i\left(1+T\left(\tilde{a}_i\right)\right)}{\sum_{i=1}^{n}\omega_i\left(1+T\left(\tilde{a}_i\right)\right)}}, \tag{11}$$

with condition (8).

Especially, if $\omega = \left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right)^T$, then the FPWG operator reduces to a fuzzy power geometric (FPG) operator:

$$FPG_{\omega}\left(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n\right) = \prod_{i=1}^{n}\left(\tilde{a}_i\right)^{\frac{\left(1+T\left(\tilde{a}_i\right)\right)}{\sum_{i=1}^{n}\left(1+T\left(\tilde{a}_i\right)\right)}}, \tag{12}$$

with condition (7).

It can be easily proved that the FPWG operator has the following properties similar to the FPWA operator.

Theorem 3. (Idempotency) If $\left(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n\right) = \tilde{a}$, then:

$$FPWG_{\omega}\left(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n\right) = \tilde{a} \cdot$$

Theorem 4. (Boundedness).

$$\min_i \tilde{a}_i \leq FPWG_{\omega}\left(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n\right) \leq \max_i \tilde{a}_i \cdot$$

From the definitions of the FPWA and FPWG operators, it can be seen that the fundamental characteristics of these two operators is that they weight all the given triangular fuzzy numbers, and weighting vectors depend upon the input arguments and allow values being aggregated to support and reinforce each other. However, in many group decision making problems, in order to assign low weights to those "false" or "biased" ones to relieve the influence of unfair arguments in the decision result, all the given arguments have to be rearranged in descending or ascending order, and then the ordered positions of the input arguments should be weighed. Furthermore, the fuzzy power ordered weighted average (FPOWA) operator should be defined as follows:

$$FPOWA_w\left(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n\right)=\frac{\sum_{i=1}^{n} w_i\left(1+T\left(\tilde{a}_{\sigma(i)}\right)\right)\tilde{a}_{\sigma(i)}}{\sum_{i=1}^{n} w_i\left(1+T\left(\tilde{a}_{\sigma(i)}\right)\right)}, \tag{13}$$

where $\tilde{a}_{\sigma(i)}$ is the i^{th} largest of the triangular fuzzy sets $\left(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n\right)$, $w_i\left(i=1,2,\cdots,n\right)$ is the collection of weights such that:

$$w_i=g\left(\frac{R_i}{TV}\right)-g\left(\frac{R_{i-1}}{TV}\right), R_i=\sum_{j=1}^{i} V_{\sigma(j)}, TV=\sum_{i=1}^{n} V_{\sigma(i)}, V_{\sigma(i)}=1+T\left(\tilde{a}_{\sigma(i)}\right) \tag{14}$$

and $T\left(\tilde{a}_{\sigma(i)}\right)$ denotes the support of the i^{th} largest triangular fuzzy variable $\tilde{a}_{\sigma(i)}$ by all the other triangular fuzzy variables, i.e.:

$$T\left(\tilde{a}_{\sigma(i)}\right)=\sum_{\substack{j=1\\ j\neq i}}^{n} Sup\left(\tilde{a}_{\sigma(i)},\tilde{a}_{\sigma(j)}\right), \tag{15}$$

where $Sup\left(\tilde{a}_{\sigma(i)},\tilde{a}_{\sigma(j)}\right)$ indicates the support of j^{th} largest triangular fuzzy variable $\tilde{a}_{\sigma(i)}$ for the i^{th} largest triangular fuzzy variable $\tilde{a}_{\sigma(j)}$, and g: $\left[0,1\right]\rightarrow\left[0,1\right]$ is a basic unit-interval monotonic (BUM) function, having the properties: 1) g(0) = 0, 2) g(1) = 1, and 3) $g(x)\geq g(y), if\ x>y$.

Furthermore, we shall define a fuzzy power ordered weighted geometric (FPOWG) operator based on the FPOWA operator and the geometric mean.

$$FPWG_w\left(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n\right)=\prod_{i=1}^{n}\left(\tilde{a}_{\sigma(i)}\right)^{\frac{\omega_i\left(1+T\left(\tilde{a}_{\sigma(i)}\right)\right)}{\sum_{i=1}^{n}\omega_i\left(1+T\left(\tilde{a}_{\sigma(i)}\right)\right)}}, \tag{16}$$

where $w_i\left(i=1,2,\cdots,n\right)$ is the collection of weights satisfying the conditions (14) and (15).

2.2. FPWHA operator and FPWQA operator

Similar to WA and OWA operators (Yager 1988), weighted harmonic averaging (WHA) operator and ordered weighted harmonic averaging (OWHA) operators are introduced as follows.

Definition 5 (Bullen *et al.* 1988)**.** Let $WHA: R^{+n}\rightarrow R^{+}$, if WHA:

$$WHA_{\omega}\left(a_1,a_2,\cdots,a_n\right)=1\Big/\sum_{i=1}^{n}\frac{\omega_i}{a_i}, \tag{17}$$

then WHA is called a weighted harmonic averaging operator, where $\omega=\left(\omega_1,\omega_2,\cdots,\omega_n\right)^T$ is the weight vector of $\left(a_1,a_2,\cdots,a_n\right)$, with $\omega_i\in\left[0,1\right]$ and $\sum_{i=1}^{n}\omega_i=1$, R is the set of all positive real numbers.

Chen *et al.* (2004), Yager (2004), Merigó and Gil-Lafuente (2009) developed the ordered weighted harmonic averaging (OWHA) operator.

Definition 6. An ordered weighted harmonic averaging operator of dimension n is a mapping $OWHA:R^n\rightarrow R$ that has an associated vector $w=\left(w_1,w_2,\cdots,w_n\right)^T$ such that $w_i>0$ and $\sum_{i=1}^{n}w_i=1$. Furthermore,

$$OWHA_w\left(a_1,a_2,\cdots,a_n\right)=1\Big/\sum_{i=1}^{n}\frac{w_i}{a_{\sigma(i)}}, \tag{18}$$

where $\left(\sigma(1),\sigma(2),\cdots,\sigma(n)\right)$ is a permutation of $\left(1,2,\cdots,n\right)$, such that $\alpha_{\sigma(i-1)}\geq\alpha_{\sigma(i)}$ for all $i=2,\cdots,n$.

In the following, based on fuzzy power weighted average (FPWA) operator and harmonic averaging operator, the fuzzy power weighted harmonic averaging operator (FPWHA) should be developed, which allows the input data to support each other in the aggregating process.

Definition 7. Let $\tilde{a}_i=\left[a_i^L,a_i^M,a_i^U\right]\left(i=1,2,\cdots,n\right)$ be a set of triangular fuzzy numbers and $\omega=\left(\omega_1,\omega_2,\cdots,\omega_n\right)^T$ be the weighting vector of $\tilde{a}_i\left(i=1,2,\cdots,n\right)$ and $\omega_i\in\left[0,1\right]$, $\sum_{i=1}^{n}\omega_i=1$, then the fuzzy power weighted harmonic average operator should be defined as follows:

$$FPWHA_\omega\left(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n\right)=\frac{1}{\sum_{i=1}^{n}\frac{\frac{\omega_i\left(1+T\left(\tilde{a}_i\right)\right)}{\sum_{i=1}^{n}\omega_i\left(1+T\left(\tilde{a}_i\right)\right)}}{\tilde{a}_i}}, \tag{19}$$

with condition (8).

Furthermore, the fuzzy power ordered weighted harmonic average (FPOWHA) operator should be defined as follows:

$$FPOWHA_w\left(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n\right)=\frac{1}{\sum_{i=1}^{n}\frac{\frac{w_i\left(1+T\left(\tilde{a}_{\sigma(i)}\right)\right)}{\sum_{i=1}^{n}w_i\left(1+T\left(\tilde{a}_{\sigma(i)}\right)\right)}}{\tilde{a}_{\sigma(i)}}}, \tag{20}$$

with condition (14) and (15).

Definition 8 (Yager 2004)**.** Let $WQA: R^n \to R$, if WQA:

$$WQA_{\omega}\left(a_1,a_2,\cdots,a_n\right)=\left(\sum_{i=1}^{n}\omega_i\left(a_i\right)^2\right)^{\frac{1}{2}}. \tag{21}$$

Then WQA is called a weighted quadratic averaging operator, where $\omega=\left(\omega_1,\omega_2,\cdots,\omega_n\right)^T$ is the weight vector of $\left(a_1,a_2,\cdots,a_n\right)$, with $\omega_i\in\left[0,1\right]$ and $\sum_{i=1}^{n}\omega_i=1$, R is the set of all real numbers.

Definition 9 (Merigó, Gil-Lafuente 2009)**.** An ordered weighted quadratic averaging operator of dimension n is a mapping $OWQA: R^n \to R$ that has an associated vector $w=\left(w_1,w_2,\cdots,w_n\right)^T$ such that $w_i>0$ and $\sum_{i=1}^{n}w_i=1$. Furthermore,

$$\mathrm{OWQA}_w\left(a_1,a_2,\cdots,a_n\right)=\left(\sum_{i=1}^{n}w_i\left(a_{\sigma(i)}\right)^2\right)^{\frac{1}{2}}, \tag{22}$$

where $\left(\sigma\left(1\right),\sigma\left(2\right),\cdots,\sigma\left(n\right)\right)$ is a permutation of $\left(1,2,\cdots,n\right)$, such that $\alpha_{\sigma(i-1)}\geq\alpha_{\sigma(i)}$ for all $i=2,\cdots,n$.

In the following, based on the quadratic average operator and fuzzy power weighted average (FPWA) operator, some fuzzy power weighted quadratic average (FPWQA) operator should be developed, which allows the input data to support each other in the aggregating process.

Definition 10. Let $\tilde{a}_i=\left[a_i^L,a_i^M,a_i^U\right]\left(i=1,2,\cdots,n\right)$ be a set of triangular fuzzy numbers and $\omega=\left(\omega_1,\omega_2,\cdots,\omega_n\right)^T$ be the weighting vector of $\tilde{a}_i\left(i=1,2,\cdots,n\right)$ and $\omega_i\in\left[0,1\right]$, $\sum_{i=1}^{n}\omega_i=1$, then the fuzzy power weighted quadratic average operator will be defined as follows:

$$FPWQA_{\omega}\left(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n\right)=\sqrt{\frac{\sum_{i=1}^{n}\omega_i\left(1+T\left(\tilde{a}_i\right)\right)\left(\tilde{a}_i\right)^2}{\sum_{i=1}^{n}\omega_i\left(1+T\left(\tilde{a}_i\right)\right)}}, \tag{23}$$

with condition (8).

Furthermore, the fuzzy power ordered weighted quadratic average (FPOWQA) operator should be defined as follows:

$$FPOWQA_w\left(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n\right)=\sqrt{\frac{\sum_{i=1}^{n}w_i\left(1+T\left(\tilde{a}_{\sigma(i)}\right)\right)\left(\tilde{a}_{\sigma(i)}\right)^2}{\sum_{i=1}^{n}w_i\left(1+T\left(\tilde{a}_{\sigma(i)}\right)\right)}}, \tag{24}$$

with conditions (14) and (15).

From the definitions of FPOWA, FPOWG, FPOWHA and FPOWQA operators, it can be seen that all these operators not only depend upon the input arguments and allow values being aggregated to support and reinforce each other, but also emphasize the ordered positions of all the given arguments. Similarly, FPOWA and FPOWG, FPOWHA and FPOWQA operators have also the following properties: Commutativity, Idempotency and Boundedness.

3. Models for multiple attribute group decision making with triangular fuzzy information

In this section, the power aggregation operators should be utilised to multiple attribute group decision making.

For a multiple attribute group decision making problems with triangular fuzzy information, let $X=\{X_1,X_2,\cdots,X_m\}$ be a discrete set of alternatives, $G=\{G_1,G_2,\cdots,G_n\}$ be the set of attributes, whose weight vector is $\omega=(\omega_1,\omega_2,\cdots,\omega_n)$, with $\omega_j\geq 0$, $j=1,2,\cdots,n$, $\sum_{j=1}^{n}\omega_j=1$, and let $D=\{D_1,D_2,\cdots,D_t\}$ be the set of decision makers, whose weight vector is $\nu=(\nu_1,\nu_2,\cdots,\nu_t)\in H$, with $\nu_k\geq 0$, $k=1,2,\cdots,t$, $\sum_{k=1}^{t}\nu_k=1$. Suppose that $\tilde{A}_k=\left(\tilde{a}_{ij}^{(k)}\right)_{m\times n}=\left[\left(a_{ij}^L\right)^{(k)},\left(a_{ij}^M\right)^{(k)},\left(a_{ij}^U\right)^{(k)}\right]_{m\times n}$ is the multiple attribute group decision making matrix, where $\tilde{a}_{ij}^{(k)}$ is an attribute value, which takes the form of triangular fuzzy number, given by the decision maker $D_k\in D$, for the alternative $X_i\in X$ with respect to the attribute $G_j\in G$.

Then, we utilize the FPWA (or FPWG, FPWHA, FPWQA) operator to develop an approach to multiple attribute group decision making problems with triangular fuzzy information, which can be described as following:

Approach I:

Step 1. Normalise each attribute value $\tilde{a}_{ij}^{(k)}$ in the matrix $\tilde{A}_k$ into a corresponding element in the matrix $\tilde{R}_k=\left(\tilde{r}_{ij}^{(k)}\right)_{m\times n}\left(\tilde{r}_{ij}^{(k)}=\left[\left(r_{ij}^L\right)^{(k)},\left(r_{ij}^M\right)^{(k)},\left(r_{ij}^U\right)^{(k)}\right]\right)$ using the following equations:

$$\begin{cases}\left(r_{ij}^L\right)^{(k)}=\left(a_{ij}^L\right)^{(k)}\Big/\sum_{i=1}^{m}\left(a_{ij}^U\right)^{(k)}\\ \left(r_{ij}^M\right)^{(k)}=\left(a_{ij}^M\right)^{(k)}\Big/\sum_{i=1}^{m}\left(a_{ij}^M\right)^{(k)}\text{, for benefit attribute } G_j\text{,}\\ \left(r_{ij}^U\right)^{(k)}=\left(a_{ij}^U\right)^{(k)}\Big/\sum_{i=1}^{m}\left(a_{ij}^L\right)^{(k)}\end{cases}$$

$$i=1,2,\cdots,m,j=1,2,\cdots,n,k=1,2,\cdots,t\cdot \tag{25}$$

$$\begin{cases} \left(r_{ij}^{L}\right)^{(k)} = \left(1 \Big/ \left(a_{ij}^{U}\right)^{(k)}\right) \Big/ \sum_{i=1}^{m} 1 \Big/ \left(a_{ij}^{L}\right)^{(k)} \\ \left(r_{ij}^{M}\right)^{(k)} = \left(1 \Big/ \left(a_{ij}^{M}\right)^{(k)}\right) \Big/ \sum_{i=1}^{m} 1 \Big/ \left(a_{ij}^{M}\right)^{(k)} \\ \left(r_{ij}^{U}\right)^{(k)} = \left(1 \Big/ \left(a_{ij}^{L}\right)^{(k)}\right) \Big/ \sum_{i=1}^{m} 1 \Big/ \left(a_{ij}^{U}\right)^{(k)} \end{cases} \text{, for cost attribute } G_j,$$

$$i = 1,2,\cdots,m,\ j = 1,2,\cdots,n, k = 1,2,\cdots,t. \tag{26}$$

Step 2. Calculate the support measure as follows:

$$Sup\left(\tilde{r}_{ij}^{(k)}, \tilde{r}_{ij}^{(l)}\right) = 1 - d\left(\tilde{r}_{ij}^{(k)}, \tilde{r}_{ij}^{(l)}\right),\ l = 1,2,\cdots,t, \tag{27}$$

which satisfies the support conditions 1)–3) in section 2. Here we calculate $d\left(\tilde{r}_{ij}^{(k)}, \tilde{r}_{ij}^{(l)}\right)$ with distance as follows:

$$d\left(\tilde{r}_{ij}^{(k)}, \tilde{r}_{ij}^{(l)}\right) = \frac{\left|\left(r_{ij}^{L}\right)^{(k)} - \left(r_{ij}^{L}\right)^{(l)}\right| + \left|\left(r_{ij}^{M}\right)^{(k)} - \left(r_{ij}^{M}\right)^{(l)}\right| + \left|\left(r_{ij}^{U}\right)^{(k)} - \left(r_{ij}^{U}\right)^{(l)}\right|}{3},\ l = 1,2,\cdots,t. \tag{28}$$

Step 3. Utilize the weights $\nu = \left(\nu_1, \nu_2, \cdots, \nu_t\right)$ of the decision maker $D_k \left(k = 1,2,\cdots,t\right)$ to calculate the weighted support $T\left(\tilde{r}_{ij}^{(k)}\right)$ of the triangular fuzzy preference value $\tilde{r}_{ij}^{(k)}$ by the other triangular fuzzy preference value $\tilde{r}_{ij}^{(l)}$ for the preference value $r_{ij}^{(l)}$ $\left(l = 1,2,\cdots,t, \text{and } l \neq k\right)$:

$$T\left(\tilde{r}_{ij}^{(k)}\right) = \sum_{\substack{l=1 \\ l \neq k}}^{t} \nu_l Sup\left(\tilde{r}_{ij}^{(k)}, \tilde{r}_{ij}^{(l)}\right) \tag{29}$$

and calculate the weights $\eta_{ij}^{(k)} \left(k = 1,2,\cdots,t\right)$ of the triangular fuzzy preference value $\tilde{r}_{ij}^{(k)}$ $\left(k = 1,2,\cdots,t\right)$:

$$\eta_{ij}^{(k)} = \frac{\nu_k \left(1 + T\left(r_{ij}^{(k)}\right)\right)}{\sum_{k=1}^{t} \nu_k \left(1 + T\left(r_{ij}^{(k)}\right)\right)},\ k = 1,2,\cdots,t, \tag{30}$$

where: $\eta_{ij}^{(k)} \geq 0, k = 1,2,\cdots,t$, and $\sum_{k=1}^{t} \eta_{ij}^{(k)} = 1$.

Step 4. Utilize the decision information given in matrix $\tilde{R}_k$, and the FPWA operator:

$$\tilde{r}_{ij}=\left(r_{ij}^L,r_{ij}^M,r_{ij}^U\right)=FPWA\left(\tilde{r}_{ij}^{(1)},\tilde{r}_{ij}^{(2)},\cdots,\tilde{r}_{ij}^{(t)}\right)=$$

$$\sum_{k=1}^{t}\frac{\nu_k\left(1+T\left(\tilde{r}_{ij}^{(k)}\right)\right)\tilde{r}_{ij}^{(k)}}{\sum_{k=1}^{t}\nu_k\left(1+T\left(\tilde{r}_{ij}^{(k)}\right)\right)}=\sum_{k=1}^{t}\eta_{ij}^{(k)}\tilde{r}_{ij}^{(k)},\ i=1,2,\cdots,m,\ j=1,2,\cdots,n \tag{31}$$

or the FPWG operator:

$$\tilde{r}_{ij}=\left(r_{ij}^L,r_{ij}^M,r_{ij}^U\right)=FPWG\left(\tilde{r}_{ij}^{(1)},\tilde{r}_{ij}^{(2)},\cdots,\tilde{r}_{ij}^{(t)}\right)=$$

$$\prod_{k=1}^{t}\left(\tilde{r}_{ij}^{(k)}\right)^{\frac{\nu_k\left(1+T\left(\tilde{r}_{ij}^{(k)}\right)\right)}{\sum_{k=1}^{t}\nu_k\left(1+T\left(\tilde{r}_{ij}^{(k)}\right)\right)}}=\prod_{k=1}^{t}\left(\tilde{r}_{ij}^{(k)}\right)^{\eta_{ij}^{(k)}},\ i=1,2,\cdots,m,\ j=1,2,\cdots,n \tag{32}$$

or the FPWHA operator:

$$\tilde{r}_{ij}=\left(r_{ij}^L,r_{ij}^M,r_{ij}^U\right)=FPWHA\left(\tilde{r}_{ij}^{(1)},\tilde{r}_{ij}^{(2)},\cdots,\tilde{r}_{ij}^{(t)}\right)=$$

$$\frac{1}{\sum_{k=1}^{t}\frac{\frac{\nu_k\left(1+T\left(\tilde{r}_{ij}^{(k)}\right)\right)}{\sum_{k=1}^{t}\nu_k\left(1+T\left(\tilde{r}_{ij}^{(k)}\right)\right)}}{\tilde{r}_{ij}^{(k)}}}=\frac{1}{\sum_{k=1}^{t}\frac{\eta_{ij}^{(k)}}{\tilde{r}_{ij}^{(k)}}},\ i=1,2,\cdots,m,\ j=1,2,\cdots,n \tag{33}$$

or the FPWQA operator:

$$\tilde{r}_{ij}=\left(r_{ij}^L,r_{ij}^M,r_{ij}^U\right)=FPWQA\left(\tilde{r}_{ij}^{(1)},\tilde{r}_{ij}^{(2)},\cdots,\tilde{r}_{ij}^{(t)}\right)=$$

$$\sqrt{\sum_{k=1}^{t}\frac{\nu_k\left(1+T\left(\tilde{r}_{ij}^{(k)}\right)\right)\left(\left(\tilde{r}_{ij}^{(k)}\right)\right)^2}{\sum_{k=1}^{t}\nu_k\left(1+T\left(\tilde{r}_{ij}^{(k)}\right)\right)}}=\sqrt{\sum_{k=1}^{t}\eta_{ij}^{(k)}\left(\left(\tilde{r}_{ij}^{(k)}\right)\right)^2},\ i=1,2,\cdots,m,\ j=1,2,\cdots,n \tag{34}$$

to aggregate all the individual decision matrices $\tilde{R}_k\left(k=1,2,\cdots,t\right)$ into the collective decision matrix $\tilde{R}=\left(\tilde{r}_{ij}\right)_{m\times n}=\left[a_{ij}^L,a_{ij}^M,a_{ij}^U\right]_{m\times n}$, where $\nu=\left\{\nu_1,\nu_2,\cdots,\nu_t\right\}$ is the weighting vector of decision makers.

Step 5. Aggregate all triangular fuzzy preference value $\tilde{r}_{ij}\left(j=1,2,\cdots,n\right)$ by using the fuzzy weighted average (FWA) operator:

$$\tilde{r}_i=\left(r_i^L,r_i^M,r_i^M\right)=FWA_\omega\left(\tilde{r}_{i1},\tilde{r}_{i2},\cdots,\tilde{r}_{in}\right)=\sum_{j=1}^{n}\omega_j\tilde{r}_{ij},\ i=1,2,\cdots,m, \tag{35}$$

or the fuzzy weighted geometric (FWG) operator:

$$\tilde{r}_i=\left(r_i^L,r_i^M,r_i^M\right)=FWG_\omega\left(\tilde{r}_{i1},\tilde{r}_{i2},\cdots,\tilde{r}_{in}\right)=\prod_{j=1}^{n}\tilde{r}_{ij}^{\;\omega_j},\ i=1,2,\cdots,m, \quad (36)$$

or the fuzzy linguistic weighted harmonic average (FWHA) operator:

$$\tilde{r}_i=\left(r_i^L,r_i^M,r_i^M\right)=FWHA_\omega\left(\tilde{r}_{i1},\tilde{r}_{i2},\cdots,\tilde{r}_{in}\right)=\frac{1}{\sum_{j=1}^{n}\frac{\omega_j}{\tilde{r}_{ij}}},\ i=1,2,\cdots,m, \quad (37)$$

or the fuzzy weighted quadratic average (FWQA) operator:

$$\tilde{r}_i=\left(r_i^L,r_i^M,r_i^M\right)=FWQA_\omega\left(\tilde{r}_{i1},\tilde{r}_{i2},\cdots,\tilde{r}_{in}\right)=\sqrt{\sum_{j=1}^{n}\omega_j\left(\tilde{r}_{ij}\right)^2},\ i=1,2,\cdots,m, \quad (38)$$

to derive the overall triangular fuzzy preference values $\tilde{r}_i\left(i=1,2,\cdots,m\right)$ of the alternative A_i, where $\omega=\left(\omega_1,\omega_2,\cdots,\omega_n\right)$ is the weighting vector of the attributes.

Step 6. To rank these collective overall preference values $\tilde{r}_i\left(i=1,2,\cdots,m\right)$, there should be first compared each $\tilde{r}_i$ with all the $\tilde{r}_j\left(j=1,2,\cdots,m\right)$ by using Eqn (2). For simplicity, let $p_{ij}=p\left(\tilde{r}_i\geq\tilde{r}_j\right)$, then a complementary matrix will be developed as $P=\left(p_{ij}\right)_{m\times m}$, where $p_{ij}\geq 0,\ p_{ij}+p_{ji}=1,\ p_{ii}=0.5,\ i,j=1,2,\cdots,n.$

Summing all the elements in each line of matrix P:

$$p_i=\sum_{j=1}^{m}p_{ij},i=1,2,\cdots,m. \quad (39)$$

Then the collective overall preference values are ranked $\tilde{r}_i\left(i=1,2,\cdots,m\right)$ in descending order in accordance with the values of $p_i\left(i=1,2,\cdots,m\right)$.

Step 7. Rank all the alternatives $X_i\left(i=1,2,\cdots,m\right)$ and select the best one(s) in accordance with the collective overall preference values $\tilde{r}_i\left(i=1,2,\cdots,m\right)$.

If the information about the weights of decision makers is unknown, then the FPOWA (or FPOWG, FPOWHA, FPOWQA) operator should be utilised to develop an approach to multiple attribute group decision making problems with triangular fuzzy information, which involves the following steps:

Approach II:

Step 1. See Approach I.

Step 2. Calculate the support measure as follows:

$$Sup\left(\tilde{r}_{ij}^{\sigma(k)},\tilde{r}_{ij}^{\sigma(l)}\right)=1-d\left(\tilde{r}_{ij}^{\sigma(k)},\tilde{r}_{ij}^{\sigma(l)}\right)=$$

$$1-\frac{\left|\left(r_{ij}^L\right)^{\sigma(k)}-\left(r_{ij}^L\right)^{\sigma(l)}\right|+\left|\left(r_{ij}^M\right)^{\sigma(k)}-\left(r_{ij}^M\right)^{\sigma(l)}\right|+\left|\left(r_{ij}^U\right)^{\sigma(k)}-\left(r_{ij}^U\right)^{\sigma(l)}\right|}{3}, \quad (40)$$

which indicates the support of l^{th} largest triangular fuzzy preference value $r_{ij}^{(l)}$ for the k^{th} largest triangular fuzzy preference value $\tilde{r}_{ij}^{(k)}$ of $\tilde{r}_{ij}^{(k)}\left(k=1,2,\cdots,t\right)$.

Step 3. Calculate the support $T\left(\tilde{r}_{ij}^{(k)}\right)$ of the k^{th} largest triangular fuzzy preference value $\tilde{r}_{ij}^{(k)}$ by the other triangular fuzzy preference value $\tilde{r}_{ij}^{(l)}$ $\left(l=1,2,\cdots,t, \text{and } l \neq k\right)$:

$$T\left(\tilde{r}_{ij}^{\sigma(k)}\right)=\sum_{\substack{l=1 \\ l \neq k}}^{t} Sup\left(\tilde{r}_{ij}^{\sigma(k)}, \tilde{r}_{ij}^{\sigma(l)}\right) \tag{41}$$

and utilise (14) to calculate the weights $\omega_{ij}^{(k)}\left(k=1,2,\cdots,t\right)$ associated with the k^{th} largest triangular fuzzy preference value $\tilde{r}_{ij}^{(k)}$, where:

$$\omega_{ij}^{(k)}=g\left(\frac{Q_{ij}^{(k)}}{TV_{ij}}\right)-g\left(\frac{Q_{ij}^{(k-1)}}{TV_{ij}}\right), \tag{42}$$

$$Q_{ij}^{(k)}=\sum_{l=1}^{k} \eta_{ij}^{\sigma(l)}, TV_{ij}=\sum_{l=1}^{t} \eta_{ij}^{\sigma(l)}, \eta_{ij}^{\sigma(l)}=1+T\left(\tilde{r}_{ij}^{\sigma(l)}\right), \tag{43}$$

where $\omega_{ij}^{(k)} \geq 0$, $k=1,2,\cdots,t$, and $\sum_{k=1}^{t} \omega_{ij}^{(k)}=1$.

Step 4. Utilise FPOWA operator:

$$\tilde{r}_{ij}=\left(r_{ij}^{L}, r_{ij}^{M}, r_{ij}^{U}\right)=FPOWA\left(\tilde{r}_{ij}^{(1)}, \tilde{r}_{ij}^{(2)}, \cdots, \tilde{r}_{ij}^{(t)}\right)=\sum_{k=1}^{t} \omega_{ij}^{(k)} \tilde{r}_{ij}^{(k)},$$
$$i=1,2,\cdots,m, j=1,2,\cdots,n, \tag{44}$$

or the FPOWG operator:

$$\tilde{r}_{ij}=\left(r_{ij}^{L}, r_{ij}^{M}, r_{ij}^{U}\right)=FPOWG\left(\tilde{r}_{ij}^{(1)}, \tilde{r}_{ij}^{(2)}, \cdots, \tilde{r}_{ij}^{(t)}\right)=\prod_{k=1}^{t}\left(\tilde{r}_{ij}^{(k)}\right)^{\omega_{ij}^{(k)}},$$
$$i=1,2,\cdots,m, j=1,2,\cdots,n, \tag{45}$$

or the FPOWHA operator:

$$\tilde{r}_{ij}=\left(r_{ij}^{L}, r_{ij}^{M}, r_{ij}^{U}\right)=FPOWHA\left(\tilde{r}_{ij}^{(1)}, \tilde{r}_{ij}^{(2)}, \cdots, \tilde{r}_{ij}^{(t)}\right)=\frac{1}{\sum_{k=1}^{t} \frac{\omega_{ij}^{(k)}}{\tilde{r}_{ij}^{(k)}}},$$
$$i=1,2,\cdots,m, j=1,2,\cdots,n, \tag{46}$$

or the FPOWQA operator:

$$\tilde{r}_{ij}=\left(r_{ij}^{L}, r_{ij}^{M}, r_{ij}^{U}\right)=FPOWQA\left(\tilde{r}_{ij}^{(1)}, \tilde{r}_{ij}^{(2)}, \cdots, \tilde{r}_{ij}^{(t)}\right)=\sqrt{\sum_{k=1}^{t} \omega_{ij}^{(k)}\left(\tilde{r}_{ij}^{(k)}\right)^{2}},$$
$$i=1,2,\cdots,m, j=1,2,\cdots,n, \tag{47}$$

to aggregate all the individual decision matrices $\tilde{R}_k\left(k=1,2,\cdots,t\right)$ into the collective decision matrix $\tilde{R}=\left(\tilde{r}_{ij}\right)_{m \times n}$.

Step 5. See Approach I.

Step 6. See Approach I.

Step 7. See Approach I.

In this section, we have proposed four approaches to solve the triangular fuzzy multiple attribute group decision making problems with the known weights or completely unknown weights information of decision makers. All these approaches can take into account the information about the relationships among the triangular fuzzy arguments being aggregated sufficiently, and can relieve the influence of outlier triangular fuzzy arguments on the decision result by assigning lower weights to those outliers and thus make the decision result more reflective of the total collection of arguments.

4. Numerical example

Let us suppose there is an investment company, which wants to invest a sum of money in the best option (adapted from Herrera, Herrera-Viedma 2000). There is a panel with five possible alternatives to invest the money: (1) A_1 is a car company; (2) A_2 is a food company; (3) A_3 is a computer company; (4) A_4 is an arms company; (5) A_5 is a TV company. The investment company must take a decision according to the following four attributes: (1) G_1 is the risk analysis; (2) G_2 is the growth analysis; (3) G_3 is the social-political impact analysis; (4) G_4 is the environmental impact analysis. The five possible alternatives $X_i\left(i=1,2,3,4,5\right)$ are to be evaluated using the triangular fuzzy numbers by the three decision makers $D_k\left(k=1,2,3\right)$ (whose weighting vector is $\nu=\left(0.4,0.3,0.3\right)$) under the above four attributes (whose weighting vector is $\omega=\left(0.3,0.1,0.2,0.4\right)$), and construct, respectively, the triangular fuzzy decision matrices are shown in Tables 1–3:

Table 1. Decision matrix $\tilde{A}_1$

	G_1	G_2	G_3	G_4
X_1	(0.53,0.55,0.58)	(0.24,0.27,0.30)	(0.42,0.47,0.52)	(0.34,0.38,0.42)
X_2	(0.22,0.29,0.32)	(0.31,0.34,0.37)	(0.50,0.52,0.55)	(0.29,0.39,0.45)
X_3	(0.39,0.41,0.44)	(0.55,0.57,0.58)	(0.57,0.59,0.60)	(0.52,0.55,0.57)
X_4	(0.60,0.61,0.62)	(0.59,0.61,0.62)	(0.44,0.47,0.50)	(0.59,0.60,0.62)
X_5	(0.45,0.47,0.50)	(0.56,0.59,0.61)	(0.40,0.41,0.43)	(0.40,0.41,0.43)

Table 2. Decision matrix $\tilde{A}_2$

	G_1	G_2	G_3	G_4
X_1	(0.76,0.78,0.81)	(0.47,0.50,0.53)	(0.65,0.70,0.75)	(0.57,0.61,0.65)
X_2	(0.45,0.52,0.50)	(0.54,0.57,0.60)	(0.73,0.75,0.78)	(0.52,0.62,0.68)
X_3	(0.62,0.64,0.67)	(0.78,0.80,0.81)	(0.80,0.82,0.83)	(0.75,0.78,0.80)
X_4	(0.83,0.84,0.85)	(0.82,0.84,0.85)	(0.67,0.70,0.73)	(0.82,0.83,0.85)
X_5	(0.68,0.70,0.76)	(0.79,0.82,0.80)	(0.63,0.64,0.66)	(0.63,0.64,0.68)

Table 3. Decision matrix $\tilde{A}_3$

	G_1	G_2	G_3	G_4
X_1	(0.62,0.65,0.68)	(0.72,0.76,0.80)	(0.91,0.93,0.96)	(0.80,0.85,0.90)
X_2	(0.69,0.72,0.75)	(0.67,0.77,0.83)	(0.60,0.67,0.70)	(0.88,0.90,0.93)
X_3	(0.93,0.95,0.96)	(0.90,0.93,0.95)	(0.77,0.79,0.82)	(0.95,0.97,0.98)
X_4	(0.97,0.99,1.00)	(0.97,0.98,1.00)	(0.98,0.99,1.00)	(0.82,0.85,0.88)
X_5	(0.94,0.97,0.99)	(0.78,0.79,0.81)	(0.83,0.85,0.88)	(0.78,0.79,0.81)

Since the weights of the decision makers are known, we shall utilise Approach I to select the most desirable alternative(s):

Step 1. Constructing the normalized decision matrix $\tilde{R}_k$. The results are shown in Tables 4–6.

Table 4. Decision matrix $\tilde{R}_1$

	G_1	G_2	G_3	G_4
X_1	(0.134,0.158,0.176)	(0.097,0.113,0.133)	(0.162,0.191,0.223)	(0.137,0.163,0.196)
X_2	(0.243,0.301,0.423)	(0.125,0.143,0.164)	(0.192,0.211,0.236)	(0.116,0.167,0.210)
X_3	(0.176,0.213,0.239)	(0.222,0.239,0.258)	(0.219,0.240,0.258)	(0.209,0.236,0.266)
X_4	(0.125,0.143,0.155)	(0.238,0.256,0.276)	(0.169,0.191,0.215)	(0.237,0.258,0.290)
X_5	(0.155,0.185,0.207)	(0.226,0.248,0.271)	(0.154,0.167,0.185)	(0.161,0.176,0.201)

Table 5. Decision matrix $\tilde{R}_2$

	G_1	G_2	G_3	G_4
X_1	(0.158,0.174,0.182)	(0.138,0.142,0.156)	(0.173,0.194,0.216)	(0.173,0.175,0.198)
X_2	(0.256,0.260,0.308)	(0.159,0.161,0.176)	(0.195,0.208,0.224)	(0.158,0.178,0.207)
X_3	(0.191,0.212,0.223)	(0.229,0.227,0.238)	(0.213,0.227,0.239)	(0.228,0.224,0.243)
X_4	(0.150,0.161,0.167)	(0.241,0.238,0.250)	(0.179,0.194,0.210)	(0.249,0.239,0.258)
X_5	(0.168,0.193,0.204)	(0.232,0.232,0.235)	(0.168,0.177,0.190)	(0.191,0.184,0.207)

Table 6. Decision matrix $\tilde{R}_3$

	G_1	G_2	G_3	G_4
X_1	(0.236,0.256,0.275)	(0164,0.180,0.198)	(0.209,0.220,0.235)	(0.178,0.195,0.213)
X_2	(0.214,0.231,0.247)	(0.153,0.182,0.205)	(0.138,0.158,0.171)	(0.196,0.206,0.220)
X_3	(0.167,0.175,0.184)	(0.205,0.220,0.235)	(0.177,0.187,0.200)	(0.211,0.222,0.232)
X_4	(0.160,0.168,0.176)	(0.221,0.232,0.248)	(0.225,0.234,0.244)	(0.182,0.195,0.208)
X_5	(0.162,0.171,0.182)	(0.178,0.187,0.200)	(0.190,0.201,0.215)	(0.173,0.181,0.191)

Step 2. Utilise (26)–(28) to calculate the weight $\eta_{ij}^{(k)}\left(i=1,2,3,4,5, j=1,2,3,4, k=1,2,3\right)$ associated with the attribute values $\tilde{r}_{ij}^{(k)}\left(i=1,2,3,4,5, j=1,2,3,4, k=1,2,3\right)$, which are expressed in the matrices $\eta^{(k)}=\left(\eta_{ij}^{(k)}\right)_{5\times 4}\left(k=1,2,3\right)$ which are given in Tables 7–9, respectively.

Table 7. Weight matrix $\eta^{(1)}$

	G_1	G_2	G_3	G_4
X_1	0.3872	0.3856	0.3861	0.3856
X_2	0.3849	0.3854	0.3867	0.3859
X_3	0.3862	0.3853	0.3861	0.3853
X_4	0.3851	0.3851	0.3864	0.3861
X_5	0.3859	0.3860	0.3858	0.3856

Table 8. Weight matrix $\eta^{(2)}$

	G_1	G_2	G_3	G_4
X_1	0.3095	0.3086	0.3078	0.3079
X_2	0.3088	0.3080	0.3085	0.3081
X_3	0.3078	0.3076	0.3084	0.3075
X_4	0.3077	0.3077	0.3081	0.3087
X_5	0.3073	0.3085	0.3080	0.3070

Table 9. Weight matrix $\eta^{(3)}$

	G_1	G_2	G_3	G_4
X_1	0.3033	0.3058	0.3062	0.3066
X_2	0.3062	0.3067	0.3047	0.3060
X_3	0.3060	0.3071	0.3055	0.3072
X_4	0.3071	0.3073	0.3054	0.3052
X_5	0.3068	0.3055	0.3062	0.3074

Step 3. Utilising the FPWA (or FPWG, FPWHA, FPWQA) operator to aggregate all the individual decision matrices into the collective decision matrix, the aggregating results are shown in Tables 10–13.

Table 10. Decision matrix $\tilde{R}$ (FPWA)

	G_1	G_2	G_3	G_4
X_1	(0.172,0.193,0.208)	(0130,0.142,0.160)	(0.180,0.201,0.224)	(0.160,0.177,0.202)
X_2	(0.238,0.267,0.334)	(0.144,0.161,0.181)	(0.176,0.194,0.213)	(0.153,0.183,0.212)
X_3	(0.178,0.201,0.217)	(0.219,0.230,0.245)	(0.204,0.220,0.234)	(0.215,0.228,0.249)
X_4	(0.144,0.156,0.165)	(0.234,0.243,0.259)	(0.189,0.205,0.222)	(0.224,0.233,0.255)
X_5	(0.161,0.184,0.198)	(0.213,0.224,0.238)	(0.169,0.180,0.195)	(0.174,0.180,0.200)

Table 11. Decision matrix $\tilde{R}$ (FPWG)

	G_1	G_2	G_3	G_4
X_1	(0.167,0.188,0.204)	(0127,0.140,0.158)	(0.179,0.200,0.224)	(0.159,0.176,0.202)
X_2	(0.237,0.265,0.326)	(0.143,0.160,0.180)	(0.174,0.193,0.211)	(0.150,0.182,0.212)
X_3	(0.178,0.200,0.216)	(0.219,0.229,0.245)	(0.203,0.218,0.233)	(0.215,0.228,0.248)
X_4	(0.143,0.156,0.165)	(0.234,0.243,0.259)	(0.188,0.204,0.222)	(0.222,0.231,0.253)
X_5	(0.161,0.183,0.198)	(0.212,0.223,0.237)	(0.169,0.180,0.195)	(0.174,0.180,0.200)

Table 12. Decision matrix $\tilde{R}$ (FPWHA)

	G_1	G_2	G_3	G_4
X_1	(0.163,0.185,0.200)	(0.124,0.137,0.156)	(0178,0.200,0.224)	(0.158,0.176,0.201)
X_2	(0.237,0.264,0.317)	(0.142,0.159,0.179)	(0.172,0.191,0.209)	(0.146,0.181,0.212)
X_3	(0.177,0.199,0.215)	(0.219,0.229,0.244)	(0.203,0.217,0.232)	(0.215,0.228,0.248)
X_4	(0.142,0.155,0.165)	(0.233,0.243,0.258)	(0.186,0.203,0.221)	(0.220,0.229,0.250)
X_5	(0.161,0.183,0.198)	(0.210,0.221,0.235)	(0.168,0.179,0.195)	(0.173,0.180,0.200)

Table 13. Decision matrix $\tilde{R}$ (FPWQA)

	G_1	G_2	G_3	G_4
X_1	(0.178,0.197,0.213)	(0.133,0.145,0.162)	(0.181,0.201,0.224)	(0.162,0.177,0.202)
X_2	(0.238,0.268,0.342)	(0.145,0.161,0.182)	(0.178,0.196,0.214)	(0.157,0.183,0.212)
X_3	(0.178,0.201,0.218)	(0.219,0.230,0.245)	(0.205,0.221,0.235)	(0.216,0.228,0.249)
X_4	(0.145,0.157,0.165)	(0.234,0.243,0.259)	(0.191,0.206,0.223)	(0.226,0.234,0.257)
X_5	(0.161,0.184,0.199)	(0.214,0.226,0.240)	(0.170,0.181,0.196)	(0.174,0.180,0.200)

Step 4. By utilising the decision information given in Tables 10–13, and FWA, FWG, FWHA and FWQA operators, and $\omega = \left(0.3, 0.1, 0.2, 0.4\right)$ is the weighting vector of the attributes, we derive the overall preference values of the alternatives. The aggregating results are shown in Table 14.

Table 14. The overall preference values of the alternatives

	FPWA and FWA	FPWG and FWG	FPWHA and FWHA	FPWQA and FWQA
X_1	(0.165,0.183,0.204)	(0.165,0.180,0.202)	(0.159,0.178,0.199)	(0.165,0.185,0.206)
X_2	(0.182,0.208,0.246)	(0.176,0.203,0.237)	(0.171,0.199,0.230)	(0.188,0.213,0.256)
X_3	(0.202,0.218,0.236)	(0.201,0.218,0.235)	(0.200,0.217, 0.233)	(0.203,0.219,0.237)
X_4	(0.194,0.205,0.222)	(0.189,0.201,0.217)	(0.184,0.197,0.212)	(0.198,0.209,0.227)
X_5	(0.173,0.186,0.202)	(0.172,0.185,0.202)	(0.171,0.184,0.201)	(0.174,0.186,0.203)

Step 5. According to the aggregating results shown in Table 14 and the expected value of triangular fuzzy variable by Eqn (2), the ordering of the alternatives are shown in Table 15. Note that > means "preferred to". As we can see, depending on the aggregation operators used, the ordering of the alternatives is the same. And the best alternative is X_3.

Table 15. Ordering of the alternatives

	Ordering
FPWA and FWA	$X_3>X_2>X_4>X_5>X_1$
FPWG and FWG	$X_3>X_2>X_4>X_5>X_1$
FPWHA and FWHA	$X_3>X_2>X_4>X_5>X_1$
FPWQA and FWQA	$X_3>X_2>X_4>X_5>X_1$

The Approach II can be also utilised to deal with triangular fuzzy multiple attribute group decision making problems where the information about the decision makers is completely unknown.

Conclusion

In this paper, based on the ideal of power aggregation, proposed eight triangular fuzzy power aggregation operators were proposed: the fuzzy power weighted average (FPWA) operator, the fuzzy power weighted geometric (FPWG) operator, fuzzy power weighted harmonic average (FPWHA), fuzzy power weighted quadratic average (FPWQA), the fuzzy power ordered weighted average (FPOWA) operator, the fuzzy power ordered weighted geometric (FPOWG) operator, fuzzy power ordered weighted harmonic average (FPOWHA) and fuzzy power ordered weighted quadratic average (FPOWQA). The prominent characteristic of these operators is that they take into account information about the relationship between the triangular fuzzy variables being aggregated. Then, these operators were utilised to develop some approaches to solve the triangular fuzzy multiple attribute group decision making problems with the known weights or completely unknown weights information of decision makers. Finally, some illustrative examples about the risk investment were given to verify the developed approach and to demonstrate its practicality and effectiveness.

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