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An Extension of the Product Integration Method to L^1 with Applications in Astrophysics

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Abstract. A Fredholm integral equation of the second kind in $L^1([a,b],\mathbb{C})$ with a weakly singular kernel is considered. Sufficient conditions are given for the existence and uniqueness of the solution. We adapt the product integration method proposed in $C^0([a,b],\mathbb{C})$ to apply it in $L^1([a,b],\mathbb{C})$, and discretize the equation. To improve the accuracy of the approximate solution, we use different iterative refinement schemes which we compare one to each other. Numerical evidence is given with an application in Astrophysics.

Keywords: Fredholm integral equation, product integration method, iterative refinement, Kolmogorov-Riesz-Fréchet theorem.

AMS Subject Classification: 45B05; 45A05; 45E99.

1 Introduction

We consider a Banach space X. Let T be the integral operator defined by

$$\forall x \in X,\ \forall s \in [a,b], \quad Tx(s) := \int_a^b L(s,t)H(s,t)x(t)dt, \tag{1.1}$$

where $(s,t) \mapsto H(s,t)$ is not smooth. For z in the resolvent set of T, $\mathrm{re}(T)$, and y in X we consider the Fredholm integral problem of the second kind

$$\text{Find } \varphi \in X \text{ s.t. } (T - zI)\varphi = y, \tag{1.2}$$

where I denotes the identity operator on X.

To approximate the solution of this equation, we define a finite rank approximation T_n of T, so that the approximate equation $(T_n - zI)\varphi_n = y$ or $(T_n - zI)\varphi_n = y_n$, where y_n is an approximation of y, be uniquely solvable and the sequence of approximate solutions φ_n converges to the exact solution φ when n tends to $+\infty$.

Among them, different classes of methods rely on a sequence of projections π_n converging pointwise to the identity operator I. For example the Galerkin operator is defined by $T_n = \pi_n T \pi_n$, the projection operator by $T_n = \pi_n T$, the Sloan operator by $T_n = T\pi_n$ and the Kulkarni operator by $T_n = T\pi_n + \pi_n T - \pi_n T \pi_n$ (see [5], [10]). These approximations of T are all ν-convergent to T (see [2]). This property ensures existence and uniqueness of φ_n, and convergence to φ.

In the case of the space $X := C^0([a,b],\mathbb{C})$ methods based upon numerical quadrature have been proposed, such as Nyström, truncated Nyström and subtraction of the singularity approximations (see [4]).

In $C^0([a,b],\mathbb{C})$, we also encounter the so-called product integration method (see [5]). In this space, the assumptions are as follows:

(H1) $L \in C^0([a,b]\times[a,b],\mathbb{C})$.

(H2) H verifies:

(H2.1) $c_H := \sup_{s\in[a,b]} \int_a^b |H(s,t)|dt$ is finite,

(H2.2) $\lim_{h\to 0} \omega_H(h) = 0$, where

$$\omega_H(h) := \sup_{|s-\tau|\le|h|,\, s,\tau\in[a,b]} \int_a^b |H(s,t) - H(\tau,t)|dt.$$

Let Δ_n, defined by

$$a =: t_{n,0} < t_{n,1} < \cdots < t_{n,n} := b \tag{1.3}$$

be a uniform grid of $[a,b]$. If $h_n := (b-a)/n$, then $t_{n,i} = a + ih_n$, for $i = 0,1,\ldots,n$. For $x \in C^0([a,b],\mathbb{C})$ and $s\in[a,b]$, the linear interpolation scheme is given by

$$[L(s,t)x(t)]_n := \frac{1}{h_n}[(t_{n,i}-t)L(s,t_{n,i-1})x(t_{n,i-1}) + (t-t_{n,i-1})L(s,t_{n,i})x(t_{n,i})]$$

for $i = 1,\ldots,n$ and $t \in [t_{n,i-1}, t_{n,i}]$.

T_n is defined by replacing $L(s,t)x(t)$ with $[L(s,t)x(t)]_n$ in (1.1). In this method T_n is a bounded finite rank linear operator defined in $C^0([a,b],\mathbb{C})$ and hence it is compact.

Under hypotheses **(H1)** and **(H2)**, for $z \in \mathrm{re}(T)$ and for n large enough, $T_n - zI$ is invertible and its inverse is uniformly bounded, (see [5]).

In this paper we extend the product integration method to the space $X := L^1([a,b],\mathbb{C})$. It will appear that the properties of the method in $C^0([a,b],\mathbb{C})$ are preserved in $L^1([a,b],\mathbb{C})$. In Section 2, we present our method and we prove the existence and uniqueness of the approximate solution and its convergence to the exact solution. Section 3 is devoted to the numerical implementation of our algorithm. The choice of the integer n is limited by the capacity of the computer. The linear system to be solved is of the order of n. So, it is interesting to improve the accuracy of the approximate solution by applying some iterative refinement schemes. Section 4 is devoted to these schemes. In Section 5, we test our approximation with an academic example. In Section 6, we apply our method to a problem belonging to Astrophysics. Our method is compared with the projection method proposed by Titaud in [1] and [11].

2 The product integration method in $L^1([a,b],\mathbb{C})$

We use the following notations: the norm in $L^1([a,b],\mathbb{C})$ is denoted by $\|x\|_1 := \int_a^b |x(s)|ds$. The subordinated operator norm is also denoted by $\|.\|_1$.

The oscillation of a function x in $L^1([a,b],\mathbb{C})$, relatively to a parameter h is defined by

$$w_1(x,h) := \sup_{|u|\in[0,|h|]} \int_a^b |x(v+u) - x(v)|dv, \tag{2.1}$$

where x is extended by 0 outside $[a,b]$.

The modulus of continuity of a continuous function on $[a,b]$ is defined as

$$w(x,h) := \sup_{u,v\in[a,b],|u-v|\leq|h|} |x(u) - x(v)|.$$

The modulus of continuity of a continuous function on $[a,b]\times[a,b]$ is defined as

$$w_2(f,h) := \sup_{u,v\in[a,b]^2,\|u-v\|\leq|h|} |f(u) - f(v)|.$$

If $x \in L^1([a,b],\mathbb{C})$, then $\lim_{h\longrightarrow 0} w_1(x,h) = 0$. If $x \in C^0([a,b],\mathbb{C})$, then $\lim_{h\longrightarrow 0} w(x,h) = 0$. If $f \in C^0([a,b]^2,\mathbb{C})$, then $\lim_{h\longrightarrow 0} w_2(f,h) = 0$.

The aim of this section is to define the approximate operator T_n. The approximate solution of (1.2) will be, if it exists and is unique, the solution φ_n of

$$(T_n - zI)\varphi_n = y. \tag{2.2}$$

T_n is constructed so that $\varphi_n \longrightarrow \varphi$. It is well known that a collectively compact convergence of T_n towards T guarantees the convergence of φ_n towards φ.

Let us recall the collectively compact convergence:

Definition 1. T_n and T are bounded linear operators from X into X. The pointwise convergence, denoted by $T_n \xrightarrow{p} T$, means that

$$\forall x \in X,\ \|T_n x - Tx\| \to 0.$$

The collectively compact convergence is denoted by $T_n \xrightarrow{cc} T$: if T is compact

$$T_n \xrightarrow{p} T$$

and for some positive integer n_0 the set

$$W := \cup_{n\geq n_0}\{T_n x\ : x \in X, \|x\| \leq 1\}$$

is relatively compact in X.

We begin by proving that T is a compact bounded linear operator from $L^1([a,b],\mathbb{C})$ into itself. Then we propose an approximate operator T_n which is a collectively compact convergent to T. Endly, we give an error estimation for the approximate solution in terms of the kernel, the norm of the exact solution, its oscillation in $L^1([a,b],\mathbb{C})$ and the mesh size.

The proof of the compactness in $L^1([a,b],\mathbb{C})$ relies on the Kolmogorov-Riesz-Fréchet theorem which is recalled here below. As usual, if A is a set of functions, we define

$$A|_\Omega := \{f|_\Omega : f \in A\},$$

where $f|_\Omega$ is the restriction of f to the subdomain Ω.

Theorem 1. *(Kolmogorov-Riesz-**Fréchet** Theorem) Let $\mathcal{F}$ be a bounded set in $L^p(\mathbb{R}^q,\mathbb{C})$, $1 \le p < \infty$. If*

$$\lim_{\|h\|\to 0} \|\tau_h f - f\|_p = 0 \tag{2.3}$$

uniformly in $f \in \mathcal{F}$, where $\tau_h f := f(.+h)$, then the closure of $\mathcal{F}|_\Omega$ is compact in $L^p(\Omega,\mathbb{C})$ for any measurable set $\Omega \subset \mathbb{R}^q$ with finite measure.

Proof. See [7]. As one finds a lot of different versions of this theorem in the litterature, we propose a proof of it in the Appendix in the case $q = 1$, $p = 1$ and $\Omega = [a,b]$. □

Now, the assumptions are as follows:

(P1) $L \in C^0([a,b]\times[a,b],\mathbb{C})$. Let

$$c_L := \sup_{(s,t)\in[a,b]^2} |L(s,t)|.$$

(P2) H verifies:

(P2.1) $c_H := \sup_{t\in[a,b]} \int_a^b |H(s,t)|ds$ is finite.

(P2.2) $\lim_{h\to 0} w_H(h) = 0$,

where

$$w_H(h) := \sup_{t\in[a,b]} \int_a^b |\tilde{H}(s+h,t) - \tilde{H}(s,t)|ds$$

and

$$\tilde{H}(s,t) := \begin{cases} H(s,t), & \text{for } s \in [a,b], \\ 0, & \text{for } s \notin [a,b]. \end{cases}$$

Lemma 1.

$$\lim_{h\to 0^+} \epsilon(H,h) = 0,$$

where

$$\epsilon(H,h) := \sup_{t\in[a,b]} \int_{b-h}^b |H(s,t)|ds.$$

Proof. For $h > 0$,

$$\begin{aligned} 0 \le \int_{b-h}^{b} |H(s,t)|ds &\le \int_{b-h}^{b} |\tilde{H}(s+h,t) - \tilde{H}(s,t)|ds \\ &\le \int_{a}^{b} |\tilde{H}(s+h,t) - \tilde{H}(s,t)|ds \le w_H(h). \end{aligned}$$

According to the assumption **(P2.2)**, $\sup_{t\in[a,b]} \int_{b-h}^{b} |H(s,t)|ds \to 0$ as $h \to 0^+$. This ends the proof. □

Theorem 2. *Under the assumptions* **(P1)** and **(P2)**, the operator T is linear from $L^1([a,b],\mathbb{C})$ into itself and compact in $L^1([a,b],\mathbb{C})$.

Proof. For all $x \in L^1([a,b],\mathbb{C})$,

$$\begin{aligned} \|Tx\|_1 &= \int_a^b |\int_a^b L(s,t)H(s,t)x(t)dt|ds \le \int_a^b \int_a^b |L(s,t)||H(s,t)||x(t)|dtds \\ &\le c_L \int_a^b |x(t)| \int_a^b |H(s,t)|dsdt \le c_L c_H \|x\|_1, \end{aligned}$$

so T is defined from $L^1([a,b],\mathbb{C})$ into itself.

The proof of the compactness of T relies on the Kolmogorov-Riesz-Fréchet theorem where $p = 1$, $q = 1$ and $\Omega = [a,b]$. We introduce the operator $\tilde{T}$:

$$\tilde{T}x(s) := \begin{cases} Tx(s), & \text{for } s \in [a,b], \\ 0, & \text{for } s \notin [a,b]. \end{cases}$$

Let A and S be the following subsets of $L^1(\mathbb{R},\mathbb{C})$ and $L^1([a,b],\mathbb{C})$ respectively:

$$\begin{aligned} A &:= \{\tilde{T}x : x \in L^1([a,b],\mathbb{C}), \|x\|_1 \le 1\}, \\ S &:= \{Tx : x \in L^1([a,b],\mathbb{C}), \|x\|_1 \le 1\}. \end{aligned}$$

A is a bounded subset of $L^1(\mathbb{R},\mathbb{C})$. Indeed

$$\|\tilde{T}x\|_1 = \|Tx\|_1 \le c_L c_H \|x\|_1 \le c_L c_H.$$

Let us prove that $\lim_{h\to 0} \|\tau_h f - f\|_1 = 0$ uniformly in $f \in A$. For $h > 0$,

$$\begin{aligned} \|\tau_h \tilde{T}x - \tilde{T}x\|_1 &= \int_a^b |\tilde{T}x(s+h) - \tilde{T}x(s)|ds \\ &= \int_a^{b-h} |Tx(s+h) - Tx(s)|ds + \int_{b-h}^{b} |Tx(s)|ds. \end{aligned}$$

Hence

$$\begin{aligned} \int_{b-h}^{b} |Tx(s)|ds &= \int_{b-h}^{b} \Big| \int_a^b L(s,t)H(s,t)x(t)\,dt \Big| ds \\ &\le c_L \|x\|_1 \epsilon(H,h) \le c_L \epsilon(H,h) \end{aligned}$$

and

$$\int_a^{b-h} |Tx(s+h) - Tx(s)|ds = \int_a^{b-h} \Big| \int_a^b \big[L(s+h,t)H(s+h,t)$$
$$- L(s,t)H(s,t)\big]x(t)\,dt\Big|ds \le \int_a^{b-h}\int_a^b |L(s+h,t)||(H(s+h,t)$$
$$- H(s,t)||x(t)|dtds + \int_a^{b-h}\int_a^b |H(s,t)||L(s+h,t) - L(s,t)||x(t)|dtds$$
$$\le c_L\|x\|_1 w_H(h) + c_H\|x\|_1 w_2(L,h) \le c_L w_H(h) + c_H w_2(L,h).$$

So

$$\|\tau_h\tilde{T}x - \tilde{T}x\|_1 \le \|x\|_1 \left(c_L w_H(h) + c_H w_2(L,h) + c_L\epsilon(H,h)\right). \tag{2.4}$$

For $h < 0$, we have similar bounds. Then $\|\tau_h f - f\|_1 \to 0$ as $h \to 0$ uniformly in $f \in A$. From the Kolmogorov-Riesz-Fréchet theorem $S = A|_{[a,b]}$ is relatively compact so T is compact. $\square$

Let us define the approximate operator T_n. Let Δ_n be the partition defined by (1.3). For $x \in L^1([a,b],\mathbb{C})$, we define the operator

$$Q_n(x,s,t) := \frac{1}{h_n}\big[(t_{n,i}-t)L(s,t_{n,i-1}) + (t-t_{n,i-1})L(s,t_{n,i})\big]\frac{1}{h_n}\int_{t_{n,i-1}}^{t_{n,i}} x(u)du$$

for $i = 1,\dots,n$ and $t \in [t_{n,i-1}, t_{n,i}]$. The approximate operator T_n is given by:

$$\forall x \in L^1([a,b],\mathbb{C}), \forall s \in [a,b],\ T_n x(s) := \int_a^b Q_n(x,s,t)H(s,t)dt, \tag{2.5}$$

which can be rewritten as

$$T_n x(s) = \sum_{i=1}^n c_{n,i} w_{n,i}(s),$$

where, for $i = 1,\dots,n$,

$$c_{n,i} := \frac{1}{h_n}\int_{t_{n,i-1}}^{t_{n,i}} x(u)\,du, \quad w_{n,i}(s) := \int_{t_{n,i-1}}^{t_{n,i}} Q_n(1,s,t)H(s,t)\,dt.$$

To prove that $T_n \xrightarrow{cc} T$, the following lemmas are needed.

Lemma 2. *For $i = 1,\dots,n$,*

$$\int_a^b |w_{n,i}(s)|ds \le h_n c_L c_H. \tag{2.6}$$

For $h \in \mathbb{R}^+$,

$$\int_{b-h}^b |w_{n,i}(s)|ds \le h_n c_L \epsilon(H,h), \tag{2.7}$$

$$\int_a^{b-h} \big|w_{n,i}(s+h) - w_{n,i}(s)\big|ds \le h_n c_H w_2(L,h) + h_n c_L w_H(h). \tag{2.8}$$

Proof. For $t \in [t_{n,i-1}, t_{n,i}]$,

$$\begin{aligned} Q_n(1,s,t) &= \frac{1}{h_n}\big[(t_{n,i}-t)L(s,t_{n,i-1}) + (t-t_{n,i-1})L(s,t_{n,i})\big], \\ |Q_n(1,s,t)| &\leq \frac{c_L}{h_n}\big[|t_{n,i}-t| + |t-t_{n,i-1}|\big] = c_L. \end{aligned}$$

Hence, by Fubini's theorem

$$\int_a^b |w_{n,i}(s)|ds \leq c_L \int_a^b \int_{t_{n,i-1}}^{t_{n,i}} |H(s,t)|\, dtds \leq c_L h_n c_H$$

$$\int_{b-h}^b |w_{n,i}(s)|ds \leq c_L \int_{b-h}^b \int_{t_{n,i-1}}^{t_{n,i}} |H(s,t)|\, dtds \leq c_L h_n \epsilon(H,h).$$

Also

$$\begin{aligned} \int_a^{b-h} \big|w_{n,i}(s+h) - w_{n,i}(s)\big| ds &\leq \int_a^{b-h} \Big|\int_{t_{n,i-1}}^{t_{n,i}} Q_n(1,s+h,t)H(s+h,t) \\ &- Q_n(1,s,t)H(s,t)dt\Big| ds \leq \int_a^{b-h}\int_{t_{n,i-1}}^{t_{n,i}} \Big|(Q_n(1,s+h,t) \\ &- Q_n(1,s,t))H(s+h,t)\Big|\, dtds + \int_a^{b-h}\int_{t_{n,i-1}}^{t_{n,i}}\Big|Q_n(1,s,t)(H(s+h,t) \\ &- H(s,t))\Big|\, dtds \leq h_n w_2(L,h) \sup_{t\in[a,b]} \int_a^b \big|H(s,t)\big|\, ds \\ &+ c_L h_n \sup_{t\in[a,b]} \int_a^{b-h} \big|\tilde{H}(s+h,t) - \tilde{H}(s,t)\big|\, ds \leq h_n c_H w_2(L,h) + h_n c_L w_H(h). \end{aligned}$$

This ends the proof. □

Lemma 3. *For $x \in L^1([a,b],\mathbb{C})$,*

$$\sum_{i=1}^n \int_{t_{n,i-1}}^{t_{n,i}} |x(u) - c_{n,i}|du \leq 2w_1(x,h_n),$$

where $w_1(x,h_n)$ is defined by (2.1). For $t \in [a,b]$,

$$\big|Q_n(1,s,t) - L(s,t)\big| \leq w_2(L,h_n).$$

Proof. For $i = 1,\dots,n$,

$$\begin{aligned} \int_{t_{n,i-1}}^{t_{n,i}} |x(u) - c_{n,i}|du &\leq \frac{1}{h_n}\int_{t_{n,i-1}}^{t_{n,i}}\int_{t_{n,i-1}}^{t_{n,i}} |x(u)-x(v)|dvdu \\ &= \frac{2}{h_n}\int_{t_{n,i-1}}^{t_{n,i}}\int_v^{t_{n,i}} |x(u)-x(v)|dudv \end{aligned}$$

$$= \frac{2}{h_n} \int_{t_{n,i-1}}^{t_{n,i}} \int_0^{t_{n,i}-v} |x(\tau+v) - x(v)| d\tau dv$$

$$\leq \frac{2}{h_n} \int_{t_{n,i-1}}^{t_{n,i}} \int_0^{h_n} |x(\tau+v) - x(v)| d\tau dv \leq \frac{2}{h_n} \int_0^{h_n} \int_{t_{n,i-1}}^{t_{n,i}} |x(\tau+v) - x(v)| dv d\tau.$$

Hence

$$\sum_{i=1}^{n} \int_{t_{n,i-1}}^{t_{n,i}} |x(u) - c_{n,i}| du \leq \frac{2}{h_n} \int_0^{h_n} \int_a^b |x(\tau+v) - x(v)| dv d\tau$$

$$\leq 2 \sup_{\tau \in [0,h_n]} \int_{t_{n,i-1}}^{t_{n,i}} |x(\tau+v) - x(v)| dv = 2w_1(x, h_n).$$

For $i = 1, \ldots, n$ and $t \in [t_{n,i-1}, t_{n,i}]$,

$$\begin{aligned}\big|Q_n(1,s,t) - L(s,t)\big| &\leq \frac{1}{h_n}\big[(t_{n,i} - t)(\big|L(s, t_{n,i-1}) - L(s,t)\big|) \\ &\quad + (t - t_{n,i-1})(\big|L(s, t_{n,i}) - L(s,t))\big|\big] \\ &\leq \sup_{s \in [a,b]} w(L(s,.), h_n) \frac{1}{h_n}\big[(t_{n,i} - t) + (t - t_{n,i-1})\big] \leq w_2(L, h_n)\end{aligned}$$

and the proof is complete. □

Theorem 3. T_n *is a compact linear operator from* $L^1([a,b], \mathbb{C})$ *into itself and*

$$T_n \xrightarrow{cc} T.$$

Proof. Due to (2.6) in Lemma 2, for $x \in L^1([a,b], \mathbb{C})$, $\|T_n x\|_1 \leq c_L c_H \|x\|_1$ so T_n is a linear bounded operator from $L^1([a,b], \mathbb{C})$ into itself. As T_n is a linear bounded operator of finite rank, it is compact. Let us prove that $T_n \xrightarrow{p} T$. Lemma 3 implies that

$$\begin{aligned}\|T_n x - Tx\|_1 &= \int_a^b \Big| \sum_{i=1}^{n} c_{n,i} \int_{t_{n,i-1}}^{t_{n,i}} Q_n(1,s,t) H(s,t) dt \\ &\quad - \int_a^b L(s,t) H(s,t) x(t) dt \Big| ds \\ &= \int_a^b \Big| \sum_{i=1}^{n} \int_{t_{n,i-1}}^{t_{n,i}} \Big(c_{n,i} Q_n(1,s,t) - L(s,t) x(t)\Big) H(s,t) dt \Big| ds \\ &= \int_a^b \Big| \sum_{i=1}^{n} \int_{t_{n,i-1}}^{t_{n,i}} \Big(\big(Q_n(1,s,t) - L(s,t)\big) x(t) \\ &\quad + Q_n(1,s,t)\big(c_{n,i} - x(t)\big)\Big) H(s,t) dt \Big| ds \\ &\leq \int_a^b \sum_{i=1}^{n} \int_{t_{n,i-1}}^{t_{n,i}} \big|Q_n(1,s,t) - L(s,t)\big| |x(t)| |H(s,t)| dt ds \\ &\quad + \int_a^b \sum_{i=1}^{n} \int_{t_{n,i-1}}^{t_{n,i}} |Q_n(1,s,t)| |c_{n,i} - x(t)| |H(s,t)| dt ds\end{aligned}$$

$$\begin{aligned} &\leq c_H\|x\|_1 w_2(L,h_n) + c_H c_L \sum_{i=1}^{n} \int_{t_{n,i-1}}^{t_{n,i}} |c_{n,i} - x(t)| dt \\ &\leq c_H\|x\|_1 w_2(L,h_n) + 2c_H c_L w_1(x,h_n). \end{aligned}$$

Hence

$$\|T_n x - Tx\|_1 \leq c_H\|x\|_1 w_2(L,h_n) + 2c_H c_L w_1(x,h_n). \tag{2.9}$$

So we have $T_n \xrightarrow{p} T$. To prove the relatively compactness of

$$S_n := \{T_n x : \; n \geq 1, \; x \in L^1([a,b],\mathbb{C}), \|x\|_1 \leq 1\}$$

we follow the same scheme as in the proof of the compactness of T. We define the operator

$$\tilde{T}_n x(s) := \begin{cases} T_n x(s), & \text{for } s \in [a,b], \\ 0, & \text{for } s \notin [a,b], \end{cases}$$

and A_n as the following subset of $L^1(\mathbb{R},\mathbb{C})$

$$A_n := \{\tilde{T}_n x : x \in L^1([a,b],\mathbb{C}), \|x\|_1 \leq 1\}.$$

A_n is a bounded subset of $L^1(\mathbb{R},\mathbb{C})$. Indeed,

$$\|\tilde{T}_n x\|_1 = \|T_n x\|_1 \leq c_L c_H \|x\|_1 \leq c_L c_H.$$

Let us prove that $\lim_{h\to 0} \|\tau_h f - f\|_1 = 0$ uniformly in $f \in A_n$. For $h > 0$,

$$\begin{aligned} \|\tau_h \tilde{T}_n x - \tilde{T}_n x\|_1 &= \int_a^b |\tilde{T}_n x(s+h) - \tilde{T}_n x(s)| ds \\ &= \int_a^{b-h} |T_n x(s+h) - T_n x(s)| ds + \int_{b-h}^{b} |T_n x(s)| ds. \end{aligned}$$

Hence, by (2.7) in Lemma 2,

$$\begin{aligned} \int_{b-h}^{b} |T_n x(s)| ds &\leq \sum_{i=1}^{n} |c_{n,i}| \int_{b-h}^{b} |w_{n,i}(s)| ds \\ &\leq \frac{1}{h_n} \|x\|_1 h_n c_L \epsilon(H,h) \leq c_L \epsilon(H,h) \end{aligned}$$

and because of (2.8) in Lemma 2,

$$\begin{aligned} &\int_a^{b-h} |T_n x(s+h) - T_n x(s)| \, ds \leq \int_a^{b-h} \sum_{i=1}^{n} \big|c_{n,i}\big| \big|w_{n,i}(s+h) - w_{n,i}(s)\big| \, ds \\ &\quad \leq \sum_{i=1}^{n} \big|c_{n,i}\big| (h_n c_H w_2(L,h) + h_n c_L w_H(h)) \\ &\quad \leq \|x\|_1 (c_H w_2(L,h) + c_L w_H(h)) \leq c_H w_2(L,h) + c_L w_H(h). \end{aligned}$$

Hence

$$\|\tau_h \tilde{T}_n x - \tilde{T}_n x\|_1 \leq \|x\|_1 \left(c_H w_2(L,h) + c_L w_H(h) + c_L \epsilon(H,h) \right). \qquad (2.10)$$

For $h < 0$, we have similar bounds. Then $\|\tau_h f - f\|_1 \to 0$ as $h \to 0$ uniformly in $f \in A_n$. From the Kolmogorov-Riesz-Fréchet theorem, $A_n|_{[a,b]}$ is relatively compact so $T_n \xrightarrow{cc} T$. □

Proposition 1. *Let $z \in \mathrm{re}(T)$. For n large enough, $T_n - zI$ is invertible and it exists a positive number $c_z > 0$ such that*

$$\|(T_n - zI)^{-1}\|_1 \leq c_z. \qquad (2.11)$$

Proof. It is a consequence of the collectively compact convergence (see [3]). □

Theorem 4. *For $z \in \mathrm{re}(T)$ and under hypotheses* **(P1)** and **(P2)**, for n large enough, the approximate operator equation (2.2) has a unique solution φ_n satisfying the following error bound:

$$\|\varphi - \varphi_n\|_1 \leq c_z c_H \left(\|\varphi\|_1 w_2(L,h_n) + 2c_L w_1(\varphi,h_n) \right).$$

Proof. According to (2.9) in the proof of Theorem 3,

$$\begin{aligned} \|\varphi - \varphi_n\| &\leq \|(T_n - zI)^{-1}\|_1 \|(T - T_n)\varphi\|_1 \\ &\leq c_z c_H \left(\|\varphi\|_1 w_2(L,h_n) + 2c_L w_1(\varphi,h_n) \right), \end{aligned}$$

which ends the proof. □

Remark 1. Often in practice, the kernel H is of convolution type. Let us fix $a = 0$ and $b = 1$. We suppose that there is a function g such that

$$H(s,t) = g(|s-t|),$$

where g is a weakly singular function defined on $]0,1]$. This means that g satisfies the following properties:

$$\lim_{s \to 0^+} g(s) = +\infty, \quad g \in C^0(]0,1],\mathbb{R}) \cap L^1([0,1],\mathbb{R}),$$
$$g \geq 0 \text{ and } g \text{ is a decreasing function in }]0,1].$$

Proposition 2. *When the factor H in the kernel of the operator T is of weakly singular convolution type, then H verifies all the conditions imposed by the product integration methods.*

Proof.

(H2.1) $\forall s \in [0,1]$, we have

$$\begin{aligned} \int_0^1 g(|s-t|)dt &= \int_0^s g(s-t)\,dt + \int_s^1 g(t-s)\,dt \\ &= \int_0^s g(\tau)d\tau + \int_0^{1-s} g(\tau)d\tau \leq 2\int_0^1 g(\tau)\,d\tau < +\infty. \end{aligned}$$

(P2.1) is also valid because the variables s and t play symmetric roles.

(H2.2) Let us prove that, for $h > 0$,

$$\lim_{h\to 0^+} \sup_{\substack{|s-\tau|\leq h\\ s,\tau\in[0,1]}} \int_0^1 |g(|s-t|) - g(|\tau - t|)|dt = 0.$$

Let ψ be the function defined by $t \mapsto \psi(t) = |g(|s-t|) - g(|\tau - t|)|$. Suppose that $\tau < s$. It is easy to prove that ψ has an axial symmetry with respect to $\xi = s + \tau/2$ over the interval $[\tau, s]$. Let $G(t) := \int_0^t g(s)ds$. Then

$$\begin{aligned}
\int_0^1 \psi(t)dt &= \int_0^\tau \psi(t)dt + \int_\tau^\xi \psi(t)dt + \int_\xi^s \psi(t)dt + \int_s^1 \psi(t)dt \\
&= \int_0^\tau g(\tau - t) - g(s-t)dt + 2\int_\tau^\xi g(t-\tau) - g(s-t)dt \\
&\quad + \int_s^1 g(t-s) - g(t-\tau)dt \\
&= G(\tau) - G(s) + G(s-\tau) + 2G(\frac{s-\tau}{2}) - 2G(s-\tau) \\
&\quad + 2G(\frac{s-\tau}{2}) + G(1-s) + G(s-\tau) - G(1-\tau) \\
&= 4\int_0^{\frac{s-\tau}{2}} g(\sigma)d\sigma - \int_\tau^s g(\sigma)d\sigma - \int_{1-s}^{1-\tau} g(\sigma)d\sigma \leq 4\int_0^{\frac{s-\tau}{2}} g(\sigma)d\sigma,
\end{aligned}$$

hence,

$$\omega_H(h) = \sup_{|s-\tau|\leq h} \int_0^1 |g(|s-t|) - g(|\tau-t|)|dt \leq 4\int_0^{\frac{h}{2}} g(\sigma)d\sigma,$$

so

$$\lim_{h\longrightarrow 0^+} \omega_H(h) = 0.$$

(P2.2) Let us prove that, for $h > 0$,

$$\lim_{h\to 0^+} \sup_{t\in[0,1]} \int_0^1 |\tilde{g}(|s+h-t|) - \tilde{g}(|s-t|)|ds = 0.$$

For $t \in [0,1]$,

$$\begin{aligned}
\int_0^1 |\tilde{g}(|s+h-t|) - \tilde{g}(|s-t|)|ds &= \int_0^1 |\tilde{g}(|t-h-s|) - \tilde{g}(|t-s|)|ds, \\
&\leq \omega_H(h),
\end{aligned}$$

so

$$\lim_{h\longrightarrow 0^+} w_H(h) = \lim_{h\longrightarrow 0^+} \int_0^1 |\tilde{g}(|s+h-t|) - \tilde{g}(|s-t|)|ds = 0,$$

which ends the proof. □

3 Iterative refinement

Recall that $z \neq 0$ because T is compact and $z \in \text{re}(T)$. Consider that the solution of (1.2) is approximated by $G_n(z)y$, where $G_n(z)$ is an approximate inverse of $T - zI$. The accuracy of $G_n(z)y$ may be improved using the following iterative refinement schemes:

$$x_n^{(0)} := G_n(z)y, \quad x_n^{(k+1)} := x^{(0)} + (I - G_n(z)(T - zI))x_n^{(k)}.$$

In [11], $G_n(z)$ has been one of the following operators:
Scheme A (Atkinson):

$$G_n(z) := R_n(z) := (T_n - zI)^{-1},$$

Scheme B (Brakhage):

$$G_n(z) := \frac{1}{z}(R_n(z)T - I),$$

Scheme C (Titaud):

$$G_n(z) := \frac{1}{z}(TR_n(z) - I).$$

Their convergence properties and error bounds have already been studied in terms of T, T_n and $R_n(z)$ (see [11] pp 40-41). If φ is the solution of (1.2),
Scheme A (Atkinson):

$$\|x_n^{(k)} - \varphi\|_1/\|\varphi\|_1 \leq \|\big(R_n(z)(T_n - T)\big)^{k+1}\|_1,$$

Scheme B (Brakhage):

$$\|x_n^{(k)} - \varphi\|_1/\|\varphi\|_1 \leq \|\big(\frac{1}{z}R_n(z)(T_n - T)T\big)^{k+1}\|_1,$$

Scheme C (Titaud):

$$\|x_n^{(k)} - \varphi\|_1/\|\varphi\|_1 \leq \|\big(\frac{1}{z}TR_n(z)(T_n - T)\big)^{k+1}\|_1.$$

Let us state error estimations for these three refinement schemes for the approximate operator T_n defined by (2.5) in this paper.

Theorem 5. *For T_n defined by (2.5), the following error bounds are satisfied:*
Scheme A (Atkinson):

$$\|x_n^{(2\ell-1)} - \varphi\|_1/\|\varphi\|_1 \leq m_z^\ell \mathcal{E}(h_n)^\ell,$$
$$\|x_n^{(2\ell)} - \varphi\|_1/\|\varphi\|_1 \leq 2d_z c_H c_L m_z^\ell \mathcal{E}(h_n)^\ell,$$

Scheme B (Brakhage):

$$\|x_n^{(2\ell-1)} - \varphi\|_1/\|\varphi\|_1 \leq \Big(d_z/z\Big)^{2\ell} \mathcal{E}(h_n)^{2\ell},$$

$$\|x_n^{(2\ell)} - \varphi\|_1/\|\varphi\|_1 \leq \Big(d_z/z\Big)^{2\ell+1} \mathcal{E}(h_n)^{2\ell+1},$$

Scheme C (Titaud):

$$\|x_n^{(2\ell-1)} - \varphi\|_1/\|\varphi\|_1 \le \big(d_z/z\big)^{2\ell}(2c_H^2c_L^2)\mathcal{E}(h_n)^{2\ell-1},$$
$$\|x_n^{(2\ell)} - \varphi\|_1/\|\varphi\|_1 \le \big(d_z z\big)^{2\ell+1}(2c_H^2c_L^2)\mathcal{E}(h_n)^{2\ell},$$

where

$$\mathcal{E}(h_n) := 3c_H^2c_Lw_2(L,h_n) + 2c_Hc_L^2w_H(h_n) + 2c_Hc_L^2\epsilon(H,h_n),$$
$$m_z := 2d_z^2 + 2c_Hc_Ld_z^3, \quad d_z := \max(c_z, \|R(z)\|).$$

Proof. Using (2.9),

$$\begin{aligned}\|(T-T_n)Tx\| &\le c_H\|Tx\|w_2(L,h_n) + 2c_Hc_Lw_1(Tx,h_n)\\ &\le c_Hw_2(L,h_n)\|T\|\|x\| + 2c_Hc_Lw_1(Tx,h_n).\end{aligned}$$

As

$$w_1(Tx,h_n) = \sup_{|u|\in[0,h_n]}\|\tau_u\tilde{T}x - \tilde{T}x\|_1$$

and due to (2.4),

$$\begin{aligned}\|(T-T_n)Tx\| &\le c_Hw_2(L,h_n)\|T\|\|x\| + 2c_Hc_L\|x\|_1(c_Lw_H(h_n)\\ &\quad + c_Hw_2(L,h_n) + c_L\epsilon(H,h_n))\\ &\le \|x\|_1\big(3c_H^2c_Lw_2(L,h_n) + 2c_Hc_L^2w_H(h_n) + 2c_Hc_L^2\epsilon(H,h_n)\big) \le \|x\|_1\mathcal{E}(h_n).\end{aligned}$$

Using (2.9),

$$\|(T-T_n)T_nx\| \le c_H\|T_nx\|w_2(L,h_n) + 2c_Hc_Lw_1(T_nx,h_n).$$

As

$$\|T_nx\|_1 \le c_Lc_H\|x\|_1, \quad w_1(T_nx,h_n) = \sup_{|u|\in[0,h_n]}\|\tau_u\tilde{T}_nx - \tilde{T}_nx\|_1$$

and because of (2.10),

$$\begin{aligned}\|(T-T_n)T_nx\| &\le c_H^2c_Lw_2(L,h_n)\|x\|_1\\ &\quad + 2c_Hc_L\|x\|_1\left(c_Lw_H(h_n) + c_Hw_2(L,h_n) + c_L\epsilon(H,h_n)\right)\\ &\le \|x\|_1\big(3c_H^2c_Lw_2(L,h_n) + 2c_Hc_L^2w_H(h_n) + 2c_Hc_L^2\epsilon(H,h_n)\big) \le \|x\|_1\mathcal{E}(h_n).\end{aligned}$$

• *Scheme A*. As

$$(T_n - T)R_n(z)T = (T_n - T)R_n(z)(T - T_n)TR_(z) + (T_n - T)TR_(z)$$

and according to (2.11),

$$\begin{aligned}\|\big(R_n(z)(T_n-T)\big)^2\| &= \|R_n(z)(T_n-T)R_n(z)T_n + R_n(z)(T_n-T)R_n(z)T\|\\ &= \|\ R_n(z)(T_n-T)T_nR_n(z) + R_n(z)(T_n-T)R_n(z)T\|\\ &\le c_z^2\|(T_n-T)T_n\| + c_z\|(T_n-T)R_n(z)(T-T_n)TR(z) + (T_n-T)TR_(z)\|.\end{aligned}$$

We have

$$\|\big(R_n(z)(T_n-T)\big)^2\| \leq d_z^2\|(T_n-T)T_n\| + 2c_Hc_Ld_z^3\|(T-T_n)T\| \\ + d_z^2\|(T_n-T)T\| \leq (2d_z^2+2c_Hc_Ld_z^3)\mathcal{E}(h_n) \leq m_z\mathcal{E}(h_n).$$

Then

$$\|\big(R_n(z)(T_n-T)\big)^{2\ell}\|_1 \quad \leq \quad m_z^\ell\mathcal{E}(h_n)^\ell,$$

so

$$\frac{\|x_n^{(2\ell-1)}-\varphi\|_1}{\|\varphi\|_1} \leq m_z^\ell\mathcal{E}(h_n)^\ell, \quad \frac{\|x_n^{(2\ell)}-\varphi\|_1}{\|\varphi\|_1} \leq 2d_zc_Hc_Lm_z^\ell\mathcal{E}(h_n)^\ell.$$

- *Scheme B.* As

$$\|\big(\frac{1}{z}R_n(z)(T_n-T)T\big)^{2\ell}\|_1 \leq \Big(\frac{d_z}{z}\Big)^{2\ell}\mathcal{E}(h_n)^{2\ell},$$

then

$$\frac{\|x_n^{(2\ell-1)}-\varphi\|_1}{\|\varphi\|_1} \leq \Big(\frac{d_z}{z}\Big)^{2\ell}\mathcal{E}(h_n)^{2\ell}, \quad \frac{\|x_n^{(2\ell)}-\varphi\|_1}{\|\varphi\|_1} \leq \Big(\frac{d_z}{z}\Big)^{2\ell+1}\mathcal{E}(h_n)^{2\ell+1}.$$

- *Scheme C.* As

$$\big(\frac{1}{z}TR_n(z)(T_n-T)\big)^{k+1} = \Big(\frac{1}{z}\Big)^{k+1}TR_n(z)\big((T_n-T)TR_n(z)\big)^k(T_n-T),$$

$$\|\Big(\frac{1}{z}TR_n(z)(T_n-T)\Big)^{k+1}\|_1 \leq \Big(\frac{d_z}{z}\Big)^{k+1}(2c_H^2c_L^2)\|(T_n-T)T\|_1^k \\ \leq \Big(\frac{d_z}{z}\Big)^{k+1}(2c_H^2c_L^2)\mathcal{E}(h_n)^k,$$

so

$$\|x_n^{(2\ell-1)}-\varphi\|_1/\|\varphi\|_1 \leq \Big(\frac{d_z}{z}\Big)^{2\ell}(2c_H^2c_L^2)\mathcal{E}(h_n)^{2\ell-1}, \\ \|x_n^{(2\ell)}-\varphi\|_1/\|\varphi\|_1 \leq \Big(\frac{d_z}{z}\Big)^{2\ell+1}(2c_H^2c_L^2)\mathcal{E}(h_n)^{2\ell}.$$

This concludes the proof. □

Remark 2. The upperbound of Scheme B appears to be the optimal one among the three error bounds. It improves slightly upon the one of Scheme C and is twice better than the one of Scheme A.

4 Numerical Implementations

The approximate equation is $T_n\varphi_n - z\varphi_n = y$, i.e.

$$\forall s \in [a,b], \quad \sum_{i=1}^{n} w_{n,j}(s)\frac{1}{h_n}\int_{t_{n,j-1}}^{t_{n,j}}\varphi_n(u)du - z\varphi_n(s) = y(s).$$

By calculating the average over $[t_{n,i-1}, t_{n,i}], i = 1, \ldots, n$, of each member of the equation, we obtain a linear system of the form $(A - zI)x = d$, where

$$\begin{aligned} A(i,j) &:= \frac{1}{h_n}\int_{t_{n,i-1}}^{t_{n,i}} w_{n,j}(s)ds, i,j = 1,\ldots,n, \\ d(i) &:= \frac{1}{h_n}\int_{t_{n,i-1}}^{t_{n,i}} y(s)ds, i = 1,\ldots,n, \\ x(i) &:= \frac{1}{h_n}\int_{t_{n,i-1}}^{t_{n,i}} \varphi_n(s)ds, i = 1,\ldots,n. \end{aligned} \tag{4.1}$$

After solving the linear system, the approximate solution can be written as

$$\varphi_n(s) = \frac{1}{z}\Big(\sum_{i=1}^{n} w_{n,j}(s)x(i) - y(s)\Big).$$

To measure the quality of the approximation we calculate the relative residual

$$r(\varphi_n) := \|(T - zI)\varphi_n - y\|_1/\|y\|_1.$$

In practice the evaluation of T is often not possible, so we replace it with T_m where $m \gg n$ and we caculate the average over $[t_{m,i-1}, t_{m,i}], i = 1, \ldots, m$, of $(T - zI)\varphi_n - y$ and of y. We obtain two vectors of size m, and we calculate the vector norm in $(\mathbb{C}^m, \|\cdot\|_1)$.

5 Numerical Illustration

As an academic example we have taken

$$-\int_0^1 \ln(|s-t|)\varphi(t)dt - \varphi(s) = y(s),$$

with unique solution $\varphi(s) = s^2$. The estimations of the relative residual with $m = 100$ for two methods: the projection method proposed by Titaud in [11] and the $L^1([a,b],\mathbb{C})$ product integration method are shown in Table 1. We observe that the $L^1([a,b],\mathbb{C})$ product integration method is faster than the projection method.

Table 1. Relative residuals.

n	Projection method	Product integration method
10	0.0968	0.0246
20	0.0499	0.0087
50	0.0211	0.0018

Figure 1 shows the profile of the matrix A defined by (4.1). It is a full matrix.

In Figure 2 we chose $n = 100$, $m = 1000$ for a relative residual tolerance of 10^{-12}. We note that Scheme B is the fastest one to reach the tolerance.

The theoretical Remark 2 of Section 3 is confirmed by this numerical experiment.

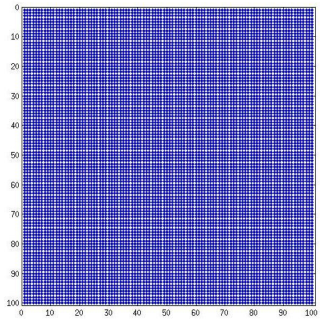

Figure 1. Matrix A of the academic illustration.

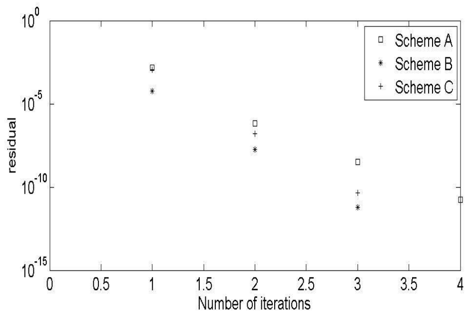

Figure 2. Residual convergence with the three refinement schemes of the academic illustration.

6 An Application in Astrophysics

The radiative transfer problem is a system of differential equations coupled with a Fredholm integral equation of the second kind. It describes the energy conserved by a beam radiation traveling, such that a beam of radiation can lose or gain energy through absorbing, scattering and emitting medium. Let τ_* be the optical width of the medium, (see [8]). An example of this equation is

$$\frac{\varpi(s)}{2}\int_0^{\tau_*} E_1(|s-t|)\varphi(t)dt - \varphi(s) = y(s),$$

where E_1 is the first integral exponential function:

$$\forall \nu \geq 1, \quad E_\nu(s) := \int_0^1 \frac{e^{-s/\mu}}{\mu^{2-\nu}} d\mu$$

and the function ϖ describes the albedo. In our numerical example $\varpi(s) = 0.7\exp(-s)$ and

$$y(s) = \begin{cases} -0.3, & \text{for } s \in [0,50[, \\ 0, & \text{for } s \in [50,100]. \end{cases}$$

The singularity of that example is different from the Cauchy singularity treated by Beltram with the product integration method in [6].

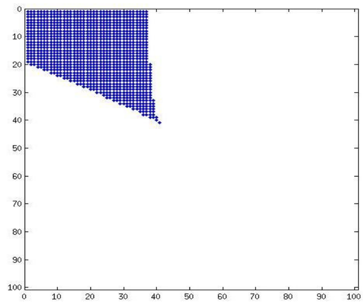

Figure 3. Matrix A of the Astrophysics application.

Figure 3 shows the profile of the matrix A defined by (4.1). It is a sparse matrix.

The relative residual associated to the approximate solution φ_n obtained by the projection method and the product integration method proposed in this paper are shown in Table 2. We observe that the product integration method converges faster than the projection method.

Table 2. Relative residuals.

n	Projection method	Product integration method
10	0.0267	0.0172
20	0.0252	0.0145
50	0.0151	0.0075

For large values of n the computation of φ_n is prohibitively costly so that we will use the refinement schemes introduced in Section s:3 to compute the final approximate solution.
In Figure 4 we chose $n = 100$, $m = 1000$ for a relative residual tolerance of 10^{-12}. We note that Scheme C is the fastest one to reach the tolerance. This confirms the results obtained in [9].

Remark 3. In this application, Scheme C is apparently faster than Scheme B. This could be explained by the difference between the profiles of the corresponding auxiliary matrices A (see Figure 1 and Figure 3).

Acknowledgements

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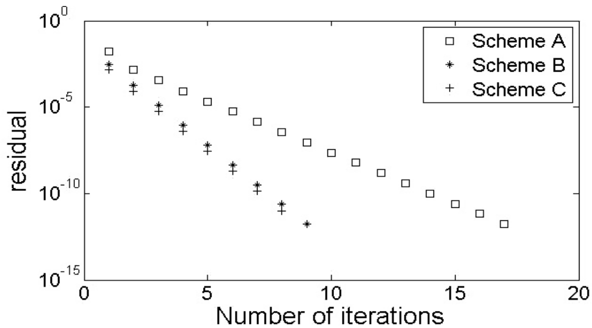

Figure 4. Residual convergence with the three refinement schemes in the Astrophysics application.

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Appendix

Proof of the Kolmogorov-Riesz-Fréchet theorem. Without loss of generality we prove the theorem for the case $p = 1$, $q = 1$ and $\Omega = [a, b]$. To simplify the notation, $\|.\|_1$ denotes the norm in $L^1(\Omega, \mathbb{C})$ and also the norm in $L^1(\mathbb{R}, \mathbb{C})$. $\|.\|_\infty$ denotes the norm in $C^0(\Omega, \mathbb{C})$ and also the norm in $C^0(\mathbb{R}, \mathbb{C})$.

As $L^1(\Omega, \mathbb{C})$ is a complete space, we just need to prove that $\mathcal{F}|_\Omega$ is precompact i.e.: For any $\varepsilon > 0$ there exist functions $f_1, f_2, \ldots, f_N \in L^1(\Omega, \mathbb{C})$ such that

$$\mathcal{F}|_\Omega \subset \cup_{i=1}^N B_1(f_i, \varepsilon),$$

where $B_1(f_i, \varepsilon)$ denotes the open ball in $L^1(\Omega, \mathbb{C})$ centered in f_i and of radius ε. The proof consists in constructing the functions f_i. The main idea of the proof is to apply a convolution regularization process to deal with continuous functions and to be able to apply the Arzela-Ascoli theorem.

Step 1: Regularization process

Let us consider the regularizing sequence defined by

$$\rho_n(x) := n\rho(nx),$$

where

$$\rho(x) := \begin{cases} k \exp(-\dfrac{1}{1 - x^2}), & \text{for } |x| \leq 1, \\ 0, & \text{otherwise,} \end{cases}$$

and k is a constant such that $\|\rho\|_1 = 1$. For all $n \in \mathbb{N}$, ρ_n is infinitely differentiable. If $*$ denotes the convolution product, and if $f \in L^1(\mathbb{R}, \mathbb{C})$, $\rho_n * f$ is a regularization of f in the sense that it is smooth: $\rho_n * f$ is infinitely differentiable. We know that $\rho_n * f \in L^1(\mathbb{R}, \mathbb{C})$ and also $\rho_n * f \longrightarrow f$ in $L^1(\mathbb{R}, \mathbb{C})$. We prove a stronger result under assumption (2.3):

$$\rho_n * f \longrightarrow f$$

uniformly in $f \in \mathcal{F}$ in $L^1(\mathbb{R}, \mathbb{C})$.

$$|\rho_n * f(x) - f(x)| \quad \leq \quad \int_{-1/n}^{1/n} |f(x - y) - f(x)|\rho_n(y)dy,$$

so that for all $f \in \mathcal{F}$,

$$\begin{aligned} \int_{\mathbb{R}} |\rho_n * f(x) - f(x)|dx &\leq \int_{\mathbb{R}} \int_{-1/n}^{1/n} |f(x - y) - f(x)|\rho_n(y)dxdy \\ &= \int_{-1/n}^{1/n} \rho_n(y)\left(\int_{\mathbb{R}} |f(x - y) - f(x)|dx\right)dy \\ &\leq \int_{-1/n}^{1/n} \rho_n(y)dy \sup_{|y| \leq \frac{1}{n}} \|\tau_y f - f\|_1 = \sup_{|y| \leq \frac{1}{n}} \|\tau_y f - f\|_1. \end{aligned}$$

Hence for all $f \in \mathcal{F}$,

$$\|\rho_n * f - f\|_1 \leq \sup_{|y|\leq 1/n} \|\tau_y f - f\|_1.$$

According to assumption (2.3), for all $\varepsilon > 0$, $\exists N_0 \in \mathbb{N}$:

$$n \geq N_0 \Rightarrow \|\rho_n * f - f\|_1 \leq \varepsilon, \text{ for all } f \in \mathcal{F}.$$

Step 2: Application of Arzela-Ascoli theorem to $H_n := \{\rho_n * f : \ f \in \mathcal{F}\}|_\Omega$

Here n is fixed. Due to the regularization properties, H_n is a subset of $C^0(\Omega, \mathbb{C})$. Let us prove that H_n is bounded in $C^0(\Omega, \mathbb{C})$ equiped with the infinity norm $\|.\|_\infty$. As $\mathcal{F}$ is bounded in $L^1(\mathbb{R}, \mathbb{C})$,

$$\|\rho_n * f\|_\infty \quad \leq \quad \|\rho_n\|_\infty \|f\|_1 \leq M\|\rho_n\|_\infty,$$

where $M := \sup_{f\in\mathcal{F}} \|f\|_1$. Let us prove that H_n is equicontinuous.

Let $x_1, x_2 \in \omega$.

$$\begin{aligned}|\rho_n * f(x_1) - \rho_n * f(x_2)| &= \Big| \int \big(\rho_n(x_1 - y) - \rho(x_2 - y)\big) f(y) dy\Big| \\ &\leq \int |\rho_n(x_1 - y) - \rho(x_2 - y)||f(y)|dy \\ &\leq \|\nabla\rho_n\|_\infty |x_1 - x_2| \, \|f\|_1 \leq M\|\nabla\rho_n\|_\infty |x_1 - x_2|,\end{aligned}$$

where $\nabla\rho_n$ is the gradient of ρ_n. According to Arzela-Ascoli theorem, H_n is relatively compact in $C^0(\Omega, \mathbb{C})$ so it is precompact.

Step 3: Construction of the functions f_i

As H_n is precompact, for $\varepsilon > 0$ there exist functions $f_i \in C^0(\Omega, \mathbb{C}), i = 1, \dots, N$, such that $H_n \subset \cup_{i=1}^N B_\infty(f_i, \varepsilon)$, where $B_\infty(f_i, \varepsilon)$ denotes the ball in $C^0(\Omega, \mathbb{C})$ centered in f_i and of radius ε, i.e:

$$\forall \rho_n * f \in H_n, \ \exists f_i \in C^0(\Omega, \mathbb{C}) : \ \|\rho_n * f - f_i\|_\infty < \varepsilon.$$

Step 4: Conclusion

Let us show that $\mathcal{F}|_\Omega$ is precompact. Let $\varepsilon > 0$ and $f \in \mathcal{F}|_\Omega$. According to the step 1, $\exists N_0 \in \mathbb{N}$:

$$n \geq N_0 \Rightarrow \|\rho_n * f - f\|_1 \leq \varepsilon, \text{ for all } f \in \mathcal{F}.$$

Let us fix $n \geq N_0$. According to the step 3, there exists $i \in \{1, \dots, N\}$, such that $\|\rho_n * f - f_i\|_\infty < \varepsilon$. We have

$$\begin{aligned}\|f - f_i\|_1 &\leq \|\rho_n * f - f\|_1 + \|\rho_n * f - f_i\|_1, \\ \|\rho_n * f - f_i\|_1 &= \left(\int_a^b \big|\rho_n * f(x) - f_i(x)\big| dx \right) \\ &\leq (b - a)\|\rho_n * f - f_i\|_\infty < (b - a)\varepsilon.\end{aligned}$$

Hence

$$\|f - f_i\|_1 \leq (1 + b - a)\varepsilon.$$

So $\mathcal{F}|_\Omega \subset \cup_{i=1}^N B_1(f_i, (1 + b - a)\varepsilon)$ and $\mathcal{F}|_\Omega$ is relatively compact.