



# A New Three-Term Conjugate Gradient-Based Projection Method for Solving Large-Scale Nonlinear Monotone Equations

Mompoti Koorapetse and Professor Kaelo

*University of Botswana*

Department of Mathematics, Private Bag UB00704, Gaborone

E-mail(*corresp.*): [kaelop@mopipi.ub.bw](mailto:kaelop@mopipi.ub.bw)

E-mail: [sphetto77@gmail.com](mailto:sphetto77@gmail.com)

Received January 22, 2019; revised September 1, 2019; accepted September 2, 2019

**Abstract.** A new three-term conjugate gradient-based projection method is presented in this paper for solving large-scale nonlinear monotone equations. This method is derivative-free and it is suitable for solving large-scale nonlinear monotone equations due to its lower storage requirements. The method satisfies the sufficient descent condition  $F_k^T d_k \leq -\tau \|F_k\|^2$ , where  $\tau > 0$  is a constant, and its global convergence is also established. Numerical results show that the method is efficient and promising.

**Keywords:** nonlinear monotone equations, derivative-free, global convergence.

**AMS Subject Classification:** 90C06; 90C30; 90C56; 65K05; 65K10.

## 1 Introduction

Conjugate gradient-based projection methods are a class of methods suited for solving large-scale nonlinear equations

$$F(x) = 0, \quad (1.1)$$

where  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a continuous and monotone function. A function  $F$  is said to be monotone if it satisfies

$$(F(x) - F(y))^T(x - y) \geq 0, \quad \forall x, y \in \mathbb{R}^n.$$

Nonlinear monotone equations arise in many practical applications, for example, as chemical equilibrium systems [13], economic equilibrium problems [5],

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and signal and image recovery problems [8]. Some mathematical problems can also be transformed into finding the solution of problem (1.1), such as variational inequality problems. This wide application has seen a number of researchers study iterative schemes (methods) for solving monotone equations over the years.

Conjugate gradient methods [3, 9, 17, 22] are very efficient methods for solving large-scale unconstrained optimization problems mainly due to their simplicity and low storage requirements. This has over the years influenced researchers to propose conjugate gradient-based projection methods by combining conjugate gradient methods with the hyperplane projection method [16] to solve nonlinear monotone equations.

Conjugate gradient-based projection methods are iterative methods that generate the next iterate  $x_{k+1}$ , given  $x_k$ , by

$$x_{k+1} = x_k - \frac{F(z_k)^T(x_k - z_k)}{\|F(z_k)\|^2} F(z_k), \tag{1.2}$$

where  $z_k = x_k + \alpha_k d_k$ ,  $\|\cdot\|$  denotes the Euclidean norm and

$$d_k = \begin{cases} -F_k, & \text{if } k = 0, \\ -F_k + \beta_k d_{k-1}, & \text{if } k \geq 1, \end{cases}$$

with  $\beta_k$  a parameter such that

$$F_k^T d_k \leq -\tau \|F_k\|^2, \quad \tau > 0, \tag{1.3}$$

$F_k = F(x_k)$  and  $\alpha_k$  is a step length. One such method is that by Papp and Rapajić [14] who proposed a derivative-free projection method

$$d_k = \begin{cases} -F_k, & \text{if } k = 0, \\ -F_k + \beta_k^{FR} w_{k-1} - \theta_k F_k, & \text{if } k \geq 1, \end{cases}$$

where

$$\beta_k^{FR} = \frac{\|F_k\|^2}{\|F_{k-1}\|^2}, \quad \theta_k = \frac{\|F_k\|^2 \|w_{k-1}\|^2}{\|F_{k-1}\|^4} \quad \text{and} \quad w_{k-1} = z_k - x_k = \alpha_k d_k.$$

The global convergence of this method was established using the line search

$$-F(z_k)^T d_k \geq \sigma \alpha_k \|F(z_k)\| \|d_k\|^2,$$

with  $\sigma > 0$  being a constant.

Another method is that by Sun and Liu [18] which proposes the search direction

$$d_k = \begin{cases} -F_k, & \text{if } k = 0, \\ -\left(1 + \beta_k \frac{F_k^T d_{k-1}}{\|F_k\|^2}\right) F_k + \beta_k d_{k-1}, & \text{if } k \geq 1, \end{cases}$$

where  $\beta_k = t \|F_k\| / \|d_{k-1}\|$ , for some positive constant  $t$ . They proved that the method is globally convergent and showed through numerical results that the method is very efficient.

More recently, Liu and Feng [10] proposed a derivative-free iterative method

$$d_k = \begin{cases} -F_k, & \text{if } k = 0, \\ -\theta_k F_k + \beta_k d_{k-1}, & \text{if } k \geq 1, \end{cases}$$

where  $\beta_k = \|F_k\|^2 / (d_{k-1}^T w_{k-1})$ , with  $w_{k-1} = y_{k-1} + t d_{k-1}$ ,  $y_{k-1} = F_k - F_{k-1}$ ,  $t = 1 + \max \left\{ 0, -\frac{d_{k-1}^T y_{k-1}}{\|d_{k-1}\|^2} \right\}$  and  $\theta_k = \tau - \frac{F_k^T d_{k-1}}{d_{k-1}^T w_{k-1}}$ , for some positive constant  $\tau$  that satisfies (1.3). This method was applied on convex constrained monotone equations. For more conjugate gradient-based projection methods, the reader is referred to [1, 2, 4, 7, 11, 12, 15, 19, 20, 21].

In this paper, motivated by the work of Zheng and Zheng [22], we propose another conjugate gradient-based projection method. This method is presented in the next section. In Section 3, we prove the global convergence of the proposed method. Numerical results follow in Section 4 and conclusion is presented in Section 5.

## 2 Algorithm

Recently, in solving an unconstrained optimization problem

$$\min_{x \in \mathbb{R}^n} f(x),$$

where  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is a continuously differentiable function, Zheng and Zheng [22] proposed two conjugate gradient methods that generate  $d_k$  by

$$d_k = \begin{cases} -g_k, & \text{if } k = 0, \\ -g_k + \beta_k d_{k-1}, & \text{if } k \geq 1, \end{cases}$$

with  $g_k = \nabla f(x_k)$  being the gradient of  $f$  at  $x_k$  and the parameter  $\beta_k$  given by

$$\beta_k^{DHS DL} = \frac{\|g_k\|^2 - \frac{\|g_k\|}{\|g_{k-1}\|} |g_k^T g_{k-1}|}{\mu |g_k^T d_{k-1}| + d_{k-1}^T y_{k-1}} - t \frac{g_k^T s_{k-1}}{d_{k-1}^T y_{k-1}},$$

$$\beta_k^{DLS DL} = \frac{\|g_k\|^2 - \frac{\|g_k\|}{\|g_{k-1}\|} |g_k^T g_{k-1}|}{\mu |g_k^T d_{k-1}| - d_{k-1}^T g_{k-1}} - t \frac{g_k^T s_{k-1}}{d_{k-1}^T y_{k-1}},$$

where  $y_{k-1} = g_k - g_{k-1}$ ,  $\mu > 1$  and  $t > 0$ . This method was shown to converge globally using the strong Wolfe line search and it was also shown to perform very well numerically.

Now, motivated the definition of  $\beta_k$  by Zheng and Zheng [22], we propose a new conjugate gradient-based projection method for (1.1) by defining the search direction as

$$d_k = \begin{cases} -F_k, & \text{if } k = 0, \\ -F_k + \beta_k d_{k-1} - \theta_k w_{k-1}, & \text{if } k \geq 1, \end{cases} \tag{2.1}$$

where

$$\beta_k = \frac{\|F_k\|^2 - \frac{\|F_k\|}{\|F_{k-1}\|} |F_k^T F_{k-1}|}{\mu \|F_k\| \|d_{k-1}\| - F_{k-1}^T d_{k-1}}, \quad \theta_k = \frac{F_k^T w_{k-1}}{\mu \|w_{k-1}\|^2}, \tag{2.2}$$

$w_{k-1} = y_{k-1} + d_{k-1}$ ,  $y_{k-1} = F_k - F_{k-1} + r s_{k-1}$ ,  $s_{k-1} = x_k - x_{k-1}$ ,  $\mu > 1$  and  $r > 0$ . Note here that  $w_{k-1}$  is defined as in [10, 11] except that in our case  $t$  is always taken to be  $t = 1$ . The specific steps of our proposed method are presented below in Algorithm 1.

*Algorithm 1* [Three-term Conjugate Gradient-based Method (TCGM)].

1. Give  $x_0 \in \mathbb{R}^n$ , the parameters  $\sigma, \kappa, r, \mu$  and  $\rho \in (0, 1)$ . Set  $k = 0$ .
2. FOR  $k = 0, 1, \dots$  do
3. If  $\|F_k\| = 0$ , then stop. Otherwise, go to Step 4.
4. Compute  $d_k$  by (2.1)–(2.2).
5. Compute  $z_k = x_k + \alpha_k d_k$  where  $\alpha_k = \max\{\kappa \rho^i : i = 0, 1, 2, \dots\}$  such that the inequality

$$-F(x_k + \alpha_k d_k)^T d_k \geq \sigma \alpha_k \|d_k\|^2 \tag{2.3}$$

with  $\sigma > 0$  and  $\kappa > 0$  being constants, is satisfied.

6. If  $\|F(z_k)\| = 0$ , then stop. Otherwise, compute  $x_{k+1}$  using (1.2).
7. Set  $k = k + 1$  and go to Step 3.
8. ENDFOR

Throughout this paper, we assume that the following assumption holds.

ASSUMPTION 1.

- (i) The function  $F(\cdot)$  is monotone on  $\mathbb{R}^n$ .
- (ii) The function  $F(\cdot)$  is Lipschitz continuous on  $\mathbb{R}^n$ , i.e. there exists a positive constant  $L$  such that

$$\|F(x) - F(y)\| \leq L \|x - y\|, \quad \forall x, y \in \mathbb{R}^n.$$

- (iii) The solution set of (1.1) is nonempty.

### 3 Convergence analysis

We now present the global convergence of our algorithm under Assumption 1.

**Lemma 1.** *Suppose that Assumption 1 holds. Let the sequence  $\{x_k\}$  be generated by Algorithm 1. Then the search direction  $d_k$  satisfies the sufficient descent condition*

$$F_k^T d_k \leq -\left(1 - \frac{1}{\mu}\right) \|F_k\|^2, \quad \forall k \geq 0 \quad \text{and} \quad \mu > 1. \tag{3.1}$$

*Proof.* Since  $d_0 = -F_0$ , we have  $F_0^T d_0 = -\|F_0\|^2$ . From this, we get that  $F_0^T d_0 \leq -\left(1 - \frac{1}{\mu}\right) \|F_0\|^2$  is satisfied for any  $\mu > 1$ . To prove (3.1) for  $k \geq 1$ , we assume  $F_{k-1}^T d_{k-1} \leq -\left(1 - \frac{1}{\mu}\right) \|F_{k-1}\|^2$  and show it is true for  $k$ . Notice from the definition of  $\beta_k$  that we have

$$0 \leq \beta_k \leq \frac{\|F_k\|^2}{\mu \|F_k\| \|d_{k-1}\|}.$$

Now, we obtain from (2.1)–(2.2) that

$$\begin{aligned} F_k^T d_k &= -\|F_k\|^2 + \beta_k F_k^T d_{k-1} - \theta_k F_k^T w_{k-1} \\ &= -\|F_k\|^2 + \frac{\|F_k\|^2 - \frac{\|F_k\|}{\|F_{k-1}\|} |F_k^T F_{k-1}|}{\mu \|F_k\| \|d_{k-1}\| - F_{k-1}^T d_{k-1}} F_k^T d_{k-1} - \frac{(F_k^T w_{k-1})^2}{\mu \|w_{k-1}\|^2} \\ &\leq -\|F_k\|^2 + \frac{\|F_k\|^2}{\mu \|F_k\| \|d_{k-1}\|} \|F_k\| \|d_{k-1}\| = -\left(1 - \frac{1}{\mu}\right) \|F_k\|^2. \end{aligned}$$

Hence (3.1) is satisfied for all  $k \geq 0$ .  $\square$

The following lemma indicates that if the sequence  $\{x_k\}$  is generated by Algorithm 1 and  $x^*$  is such that  $F(x^*) = 0$ , then the sequence  $\{x_k - x^*\}$  is decreasing and convergent, thus the sequence  $\{x_k\}$  is bounded.

**Lemma 2.** *Suppose Assumption 1 holds and the sequence  $\{x_k\}$  is generated by Algorithm 1. For any  $x^*$  such that  $F(x^*) = 0$ , we have that*

$$\|x_{k+1} - x^*\|^2 \leq \|x_k - x^*\|^2 - \|x_{k+1} - x_k\|^2$$

and the sequence  $\{x_k\}$  is bounded. Furthermore, either  $\{x_k\}$  is finite and the last iterate is a solution of (1.1) or  $\{x_k\}$  is infinite and

$$\sum_{k=0}^{\infty} \|x_{k+1} - x_k\|^2 < \infty,$$

which means

$$\lim_{k \rightarrow \infty} \|x_{k+1} - x_k\| = 0. \tag{3.2}$$

*Proof.* The conclusion follows from Theorem 2.1 in [16].  $\square$

Since  $F(x)$  is continuous and  $\{x_k\}$  is a bounded sequence it follows that there exists a constant  $M > 0$  such that  $\|F(x_k)\| \leq M, \forall k \geq 0$ . Thus,  $\{F_k\}$  is bounded.

**Lemma 3.** *For all  $k \geq 0$ , we have*

$$\left(1 - \frac{1}{\mu}\right) \|F_k\| \leq \|d_k\| \leq \left(1 + \frac{2}{\mu}\right) \|F_k\|.$$

*Proof.* From (3.1) and Cauchy-Schwarz inequality, we have

$$\|d_k\| \geq \left(1 - \frac{1}{\mu}\right) \|F_k\|.$$

From (2.1)–(2.2) we have

$$\begin{aligned} \|d_k\| &\leq \|F_k\| + |\beta_k| \|d_{k-1}\| + |\theta_k| \|w_{k-1}\| \\ &\leq \|F_k\| + \frac{\|F_k\|^2}{\mu \|d_{k-1}\| \|F_k\|} \|d_{k-1}\| + \frac{\|F_k\| \|w_{k-1}\|}{\mu \|w_{k-1}\|^2} \|w_{k-1}\| \\ &= \|F_k\| + \frac{2}{\mu} \|F_k\| = \left(1 + \frac{2}{\mu}\right) \|F_k\|. \end{aligned}$$

Therefore,

$$\left(1 - \frac{1}{\mu}\right) \|F_k\| \leq \|d_k\| \leq \left(1 + \frac{2}{\mu}\right) \|F_k\|.$$

□

**Lemma 4.** *Let Assumption 1 hold and the sequences  $\{x_k\}$  and  $\{z_k\}$  be generated by Algorithm 1. Then, we have*

$$\alpha_k \geq \min \left\{ \kappa, \frac{\rho\mu(\mu - 1)}{(L + \sigma)(\mu + 2)^2} \right\}.$$

*Proof.* If  $\alpha_k \neq \kappa$ , then  $\alpha_k' = \frac{\alpha_k}{\rho}$  does not satisfy (2.3), that is

$$-F(x_k + \alpha_k' d_k)^T d_k < \sigma \alpha_k' \|d_k\|^2.$$

This together with (2.3) and (3.1) imply that

$$\begin{aligned} \left(1 - \frac{1}{\mu}\right) \|F_k\|^2 &\leq -F_k^T d_k \\ &= (F(x_k + \alpha_k' d_k) - F_k)^T d_k - F(x_k + \alpha_k' d_k)^T d_k \\ &\leq L \alpha_k' \|d_k\|^2 + \sigma \alpha_k' \|d_k\|^2 = (L + \sigma) \alpha_k \rho^{-1} \|d_k\|^2 \\ &\leq (L + \sigma) \alpha_k \rho^{-1} \left(\frac{\mu + 2}{\mu}\right)^2 \|F_k\|^2. \end{aligned}$$

Thus  $\alpha_k \geq \rho\mu(\mu - 1)/(L + \sigma)(\mu + 2)^2$ . □

**Theorem 1.** *Suppose that Assumption 1 holds, and the sequence  $\{x_k\}$  is generated by Algorithm 1. Then we have*

$$\liminf_{k \rightarrow \infty} \|F_k\| = 0. \tag{3.3}$$

*Proof.* We assume that (3.3) does not hold, that is,  $\exists \eta > 0$  such that  $\|F_k\| \geq \eta, \forall k \geq 0$ . From (3.1) and Cauchy-Schwarz inequality, we have

$$\|d_k\| \geq (1 - 1/\mu) \|F_k\| \geq (1 - 1/\mu) \eta > 0, \quad \forall k \geq 0,$$



Problem 5. [12]

$$\begin{aligned} F_1(x) &= 2x_1 - x_2 + e^{x_1} - 1, \\ F_i(x) &= -x_{i-1} + 2x_i - x_{i+1} + e^{x_i} - 1, \quad \text{for } i = 1, 2, 3, \dots, n - 1, \\ F_n(x) &= -x_{n-1} + 2x_n + e^{x_n} - 1. \end{aligned}$$

Problem 6. [21]

$$\begin{aligned} F_{2i-1}(x) &= x_{2i-1} + ((5 - x_{2i})x_{2i} - 2)x_{2i} - 13, \\ F_{2i}(x) &= x_{2i-1} + ((1 + x_{2i})x_{2i} - 14)x_{2i} - 29, \end{aligned}$$

for  $i = 1, 2, 3, \dots, \frac{n}{2}$  ( $n$  even).

Problem 7. [21]

$$\begin{aligned} F_1(x) &= 2x_1 + 0.5h^2(x_1 + h)^3 - x_2, \\ F_i(x) &= 2x_i + 0.5h^2(x_i + ih)^3 - x_{i-1} + x_{i+1}, \quad \text{for } i = 2, 3, \dots, n - 1, \\ F_n(x) &= 2x_n + 0.5h^2(x_n + nh)^3 - x_{n-1}, \end{aligned}$$

where  $h = 1/(n + 1)$ .

Problem 8. [2]

$$F_i(x) = 2x_i - \sin(|x_i|), \quad \text{for } i = 1, 2, 3, \dots, n.$$

Problem 9. [21]

$$\begin{aligned} F_1(x) &= 3x_1^3 + 2x_2 - 5 + \sin(x_1 - x_2) \sin(x_1 + x_2), \\ F_i(x) &= -x_{i-1}e^{x_{i-1}-x_i} + x_i(4 + 3x_i^2) + 2x_{i+1} \\ &\quad + \sin(x_i - x_{i+1}) \sin(x_i + x_{i+1}) - 8, \quad \text{for } i = 2, 3, \dots, n - 1, \\ F_n(x) &= -x_{n-1}e^{x_{n-1}-x_n} + 4x_n - 3. \end{aligned}$$

Problem 10. [2]

$$\begin{aligned} F_1(x) &= 2x_1 - \sin(x_1) - 1, \\ F_i(x) &= -2x_{i-1} + 2x_i + \sin(x_i) - 1, \quad \text{for } i = 1, 2, 3, \dots, n - 1, \\ F_n(x) &= 2x_n + \sin(x_n) - 1. \end{aligned}$$

The results are presented in Tables 1–10, where  $x_0^1 = (1, 1, \dots, 1)^T$ ,  $x_0^2 = (-1, -1, \dots, -1)^T$ ,  $x_0^3 = (0.1, 0.1, \dots, 0.1)^T$  and  $x_0^4 = (-0.1, -0.1, \dots, -0.1)^T$ . In each table, we report the dimension of the problem (DIM), the number of iterations (NI), the number of function evaluations (FE) and the CPU time in seconds. We note here that all algorithms managed to solve all the ten test functions successfully. We see that the proposed method performs better than the other methods in almost all the problems.

To have comprehensive comparisons for these methods with respect to number of iterations, number of function evaluations and the CPU time, we apply the performance profiles tool of Dolan and Moré [6] to obtain Figures 1–2. The



**Table 1.** Numerical results of Problem 1.

| $x_0$   | NI    |      |        |      | FE   |        |      | CPU    |        |        |
|---------|-------|------|--------|------|------|--------|------|--------|--------|--------|
|         | DIM   | TCGM | M3TFR2 | ITDM | TCGM | M3TFR2 | ITDM | TCGM   | M3TFR2 | ITDM   |
| $x_0^1$ | 3000  | 9    | 13     | 13   | 29   | 57     | 33   | 0.0028 | 0.0066 | 0.0045 |
|         | 5000  | 9    | 18     | 13   | 29   | 85     | 33   | 0.0041 | 0.0122 | 0.0068 |
|         | 10000 | 9    | 22     | 10   | 29   | 125    | 30   | 0.0085 | 0.0340 | 0.0122 |
|         | 20000 | 9    | 28     | 10   | 29   | 191    | 30   | 0.0218 | 0.1214 | 0.0216 |
| $x_0^2$ | 3000  | 9    | 13     | 13   | 29   | 57     | 33   | 0.0029 | 0.0056 | 0.0045 |
|         | 5000  | 9    | 16     | 13   | 29   | 78     | 33   | 0.0040 | 0.0110 | 0.0069 |
|         | 10000 | 9    | 21     | 10   | 29   | 122    | 30   | 0.0108 | 0.0425 | 0.0147 |
|         | 20000 | 9    | 28     | 10   | 29   | 191    | 30   | 0.0144 | 0.1033 | 0.0249 |
| $x_0^3$ | 3000  | 8    | 16     | 13   | 20   | 80     | 34   | 0.0019 | 0.0075 | 0.0044 |
|         | 5000  | 8    | 19     | 13   | 20   | 106    | 34   | 0.0029 | 0.0148 | 0.0070 |
|         | 10000 | 8    | 22     | 14   | 20   | 149    | 37   | 0.0085 | 0.0446 | 0.0196 |
|         | 20000 | 8    | 29     | 14   | 20   | 223    | 37   | 0.0157 | 0.1245 | 0.0351 |
| $x_0^4$ | 3000  | 8    | 17     | 13   | 20   | 83     | 34   | 0.0030 | 0.0118 | 0.0048 |
|         | 5000  | 8    | 17     | 13   | 20   | 99     | 34   | 0.0029 | 0.0138 | 0.0068 |
|         | 10000 | 8    | 23     | 14   | 20   | 152    | 37   | 0.0054 | 0.0395 | 0.0138 |
|         | 20000 | 8    | 27     | 14   | 20   | 217    | 37   | 0.0104 | 0.1176 | 0.0266 |

**Table 2.** Numerical results of Problem 2.

| $x_0$   | NI   |      |        |      | FE   |        |      | CPU    |        |        |
|---------|------|------|--------|------|------|--------|------|--------|--------|--------|
|         | DIM  | TCGM | M3TFR2 | ITDM | TCGM | M3TFR2 | ITDM | TCGM   | M3TFR2 | ITDM   |
| $x_0^1$ | 300  | 17   | 17     | 22   | 45   | 58     | 72   | 0.0085 | 0.0298 | 0.0140 |
|         | 500  | 17   | 19     | 23   | 44   | 71     | 75   | 0.0296 | 0.0772 | 0.0537 |
|         | 1000 | 17   | 21     | 23   | 45   | 93     | 112  | 0.0913 | 0.1997 | 0.2398 |
|         | 2000 | 16   | 24     | 23   | 40   | 125    | 75   | 0.3556 | 1.1285 | 0.6854 |
| $x_0^2$ | 300  | 17   | 21     | 23   | 40   | 80     | 72   | 0.0069 | 0.0146 | 0.0127 |
|         | 500  | 18   | 21     | 26   | 44   | 89     | 121  | 0.0276 | 0.0561 | 0.0772 |
|         | 1000 | 21   | 22     | 25   | 58   | 109    | 80   | 0.1163 | 0.2198 | 0.1708 |
|         | 2000 | 19   | 25     | 24   | 48   | 155    | 74   | 0.4311 | 1.4178 | 0.6819 |
| $x_0^3$ | 300  | 14   | 16     | 18   | 36   | 46     | 55   | 0.0061 | 0.0087 | 0.0097 |
|         | 500  | 15   | 17     | 20   | 39   | 48     | 64   | 0.0242 | 0.0299 | 0.0411 |
|         | 1000 | 14   | 17     | 20   | 36   | 50     | 64   | 0.0707 | 0.1000 | 0.1301 |
|         | 2000 | 14   | 17     | 19   | 35   | 48     | 57   | 0.3142 | 0.4364 | 0.5250 |
| $x_0^4$ | 300  | 14   | 16     | 18   | 37   | 47     | 58   | 0.0071 | 0.0094 | 0.0113 |
|         | 500  | 14   | 16     | 19   | 36   | 47     | 61   | 0.0249 | 0.0333 | 0.0437 |
|         | 1000 | 16   | 17     | 19   | 42   | 48     | 61   | 0.0831 | 0.0970 | 0.1285 |
|         | 2000 | 15   | 17     | 19   | 38   | 50     | 61   | 0.3447 | 0.4539 | 0.5635 |

**Table 3.** Numerical results of Problem 3.

| $x_0$   | NI   |      |        |      | FE   |        |      | CPU    |        |        |
|---------|------|------|--------|------|------|--------|------|--------|--------|--------|
|         | DIM  | TCGM | M3TFR2 | ITDM | TCGM | M3TFR2 | ITDM | TCGM   | M3TFR2 | ITDM   |
| $x_0^1$ | 300  | 8    | 15     | 15   | 16   | 81     | 29   | 0.0006 | 0.0093 | 0.0014 |
|         | 500  | 8    | 18     | 16   | 16   | 108    | 31   | 0.0009 | 0.0055 | 0.0020 |
|         | 1000 | 8    | 22     | 16   | 16   | 158    | 31   | 0.0014 | 0.0138 | 0.0034 |
|         | 2000 | 8    | 29     | 16   | 16   | 242    | 31   | 0.0026 | 0.0386 | 0.0061 |
| $x_0^2$ | 300  | 8    | 29     | 16   | 16   | 207    | 31   | 0.0006 | 0.0075 | 0.0014 |
|         | 500  | 8    | 36     | 16   | 16   | 285    | 31   | 0.0009 | 0.0145 | 0.0020 |
|         | 1000 | 9    | 45     | 17   | 18   | 420    | 33   | 0.0016 | 0.0365 | 0.0036 |
|         | 2000 | 9    | 61     | 17   | 18   | 639    | 33   | 0.0029 | 0.1014 | 0.0065 |
| $x_0^3$ | 300  | 8    | 20     | 16   | 16   | 131    | 31   | 0.0006 | 0.0047 | 0.0014 |
|         | 500  | 8    | 28     | 16   | 16   | 188    | 31   | 0.0009 | 0.0096 | 0.0020 |
|         | 1000 | 8    | 33     | 16   | 16   | 276    | 31   | 0.0014 | 0.0243 | 0.0034 |
|         | 2000 | 9    | 46     | 17   | 18   | 423    | 33   | 0.0029 | 0.0673 | 0.0065 |
| $x_0^4$ | 300  | 8    | 27     | 16   | 16   | 164    | 31   | 0.0006 | 0.0060 | 0.0014 |
|         | 500  | 8    | 26     | 16   | 16   | 198    | 31   | 0.0009 | 0.0101 | 0.0020 |
|         | 1000 | 9    | 35     | 16   | 18   | 300    | 31   | 0.0016 | 0.0261 | 0.0034 |
|         | 2000 | 9    | 49     | 17   | 18   | 463    | 33   | 0.0029 | 0.0737 | 0.0065 |

**Table 4.** Numerical results of Problem 4.

| $x_0$   | NI   |      |        |      | FE   |        |      | CPU    |        |        |
|---------|------|------|--------|------|------|--------|------|--------|--------|--------|
|         | DIM  | TCGM | M3TFR2 | ITDM | TCGM | M3TFR2 | ITDM | TCGM   | M3TFR2 | ITDM   |
| $x_0^1$ | 300  | 7    | 5      | 10   | 20   | 11     | 28   | 0.0005 | 0.0030 | 0.0008 |
|         | 500  | 7    | 5      | 10   | 20   | 13     | 28   | 0.0005 | 0.0004 | 0.0010 |
|         | 1000 | 7    | 6      | 11   | 20   | 18     | 31   | 0.0007 | 0.0007 | 0.0015 |
|         | 2000 | 8    | 8      | 11   | 23   | 28     | 31   | 0.0012 | 0.0014 | 0.0022 |
| $x_0^2$ | 300  | 7    | 19     | 10   | 18   | 104    | 26   | 0.0004 | 0.0024 | 0.0008 |
|         | 500  | 7    | 21     | 11   | 18   | 135    | 29   | 0.0005 | 0.0034 | 0.0011 |
|         | 1000 | 8    | 26     | 11   | 21   | 193    | 29   | 0.0008 | 0.0062 | 0.0014 |
|         | 2000 | 8    | 35     | 11   | 21   | 295    | 29   | 0.0010 | 0.0130 | 0.0021 |
| $x_0^3$ | 300  | 8    | 13     | 10   | 22   | 46     | 27   | 0.0005 | 0.0011 | 0.0008 |
|         | 500  | 8    | 12     | 11   | 22   | 43     | 30   | 0.0006 | 0.0012 | 0.0011 |
|         | 1000 | 8    | 14     | 11   | 22   | 60     | 30   | 0.0008 | 0.0020 | 0.0014 |
|         | 2000 | 8    | 18     | 11   | 22   | 87     | 30   | 0.0011 | 0.0040 | 0.0021 |
| $x_0^4$ | 300  | 8    | 13     | 11   | 22   | 51     | 30   | 0.0005 | 0.0012 | 0.0009 |
|         | 500  | 8    | 15     | 11   | 22   | 62     | 30   | 0.0006 | 0.0016 | 0.0010 |
|         | 1000 | 8    | 16     | 12   | 22   | 78     | 33   | 0.0008 | 0.0026 | 0.0015 |
|         | 2000 | 9    | 21     | 12   | 25   | 118    | 33   | 0.0012 | 0.0053 | 0.0023 |

**Table 5.** Numerical results of Problem 5.

| $x_0$   | NI    |      |        |      | FE   |        |      | CPU    |        |        |
|---------|-------|------|--------|------|------|--------|------|--------|--------|--------|
|         | DIM   | TCGM | M3TFR2 | ITDM | TCGM | M3TFR2 | ITDM | TCGM   | M3TFR2 | ITDM   |
| $x_0^1$ | 3000  | 17   | 27     | 22   | 45   | 159    | 69   | 0.0097 | 0.0326 | 0.0165 |
|         | 5000  | 19   | 33     | 23   | 50   | 222    | 72   | 0.0142 | 0.0724 | 0.0258 |
|         | 10000 | 18   | 41     | 21   | 46   | 331    | 64   | 0.0205 | 0.1862 | 0.0343 |
|         | 20000 | 17   | 55     | 25   | 42   | 507    | 80   | 0.0416 | 0.5543 | 0.0997 |
| $x_0^2$ | 3000  | 19   | 30     | 26   | 48   | 200    | 81   | 0.0104 | 0.0447 | 0.0198 |
|         | 5000  | 19   | 32     | 25   | 47   | 252    | 78   | 0.0160 | 0.0680 | 0.0301 |
|         | 10000 | 19   | 44     | 28   | 47   | 389    | 129  | 0.0279 | 0.2348 | 0.0753 |
|         | 20000 | 17   | 57     | 26   | 40   | 577    | 83   | 0.0705 | 0.6020 | 0.1022 |
| $x_0^3$ | 3000  | 14   | 17     | 20   | 35   | 51     | 61   | 0.0051 | 0.0080 | 0.0119 |
|         | 5000  | 14   | 16     | 20   | 35   | 46     | 61   | 0.0080 | 0.0144 | 0.0213 |
|         | 10000 | 14   | 17     | 19   | 34   | 52     | 94   | 0.0153 | 0.0238 | 0.0509 |
|         | 20000 | 14   | 13     | 20   | 34   | 44     | 59   | 0.0287 | 0.0423 | 0.0603 |
| $x_0^4$ | 3000  | 15   | 18     | 15   | 38   | 53     | 44   | 0.0056 | 0.0078 | 0.0076 |
|         | 5000  | 15   | 17     | 21   | 38   | 50     | 70   | 0.0087 | 0.0115 | 0.0188 |
|         | 10000 | 16   | 18     | 19   | 41   | 58     | 59   | 0.0225 | 0.0259 | 0.0405 |
|         | 20000 | 16   | 17     | 20   | 42   | 58     | 63   | 0.0470 | 0.0496 | 0.0793 |

**Table 6.** Numerical results of Problem 6.

| $x_0$   | NI   |      |        |      | FE   |        |       | CPU    |        |        |
|---------|------|------|--------|------|------|--------|-------|--------|--------|--------|
|         | DIM  | TCGM | M3TFR2 | ITDM | TCGM | M3TFR2 | ITDM  | TCGM   | M3TFR2 | ITDM   |
| $x_0^1$ | 300  | 181  | 204    | 624  | 984  | 878    | 9849  | 0.0193 | 0.0220 | 0.1957 |
|         | 500  | 125  | 210    | 1083 | 678  | 945    | 17573 | 0.0166 | 0.0253 | 0.4370 |
|         | 1000 | 155  | 128    | 384  | 835  | 806    | 5937  | 0.0301 | 0.0305 | 0.2251 |
|         | 2000 | 231  | 196    | 881  | 1296 | 1285   | 14023 | 0.0770 | 0.0812 | 0.8993 |
| $x_0^2$ | 300  | 185  | 197    | 794  | 1030 | 895    | 12665 | 0.0205 | 0.0189 | 0.2527 |
|         | 500  | 97   | 194    | 930  | 515  | 1014   | 14747 | 0.0126 | 0.0263 | 0.3688 |
|         | 1000 | 189  | 218    | 729  | 1046 | 1365   | 11549 | 0.0374 | 0.0516 | 0.4370 |
|         | 2000 | 124  | 270    | 600  | 675  | 1874   | 9327  | 0.0400 | 0.1167 | 0.5942 |
| $x_0^3$ | 300  | 198  | 252    | 524  | 1093 | 1075   | 8202  | 0.0217 | 0.0228 | 0.1625 |
|         | 500  | 144  | 158    | 832  | 772  | 852    | 13048 | 0.0189 | 0.0221 | 0.3267 |
|         | 1000 | 154  | 242    | 845  | 842  | 1314   | 13223 | 0.0302 | 0.0504 | 0.5015 |
|         | 2000 | 193  | 229    | 680  | 1068 | 1543   | 10564 | 0.0628 | 0.0958 | 0.6666 |
| $x_0^4$ | 300  | 117  | 157    | 824  | 622  | 775    | 13077 | 0.0123 | 0.0162 | 0.2595 |
|         | 500  | 114  | 303    | 825  | 604  | 1350   | 13067 | 0.0147 | 0.0355 | 0.3267 |
|         | 1000 | 86   | 235    | 815  | 454  | 1283   | 12914 | 0.0163 | 0.0495 | 0.4895 |
|         | 2000 | 109  | 231    | 868  | 578  | 1606   | 13877 | 0.0343 | 0.0998 | 0.8735 |

**Table 7.** Numerical results of Problem 7.

| $x_0$   | NI   |      |        |      | FE   |        |      | CPU    |        |        |
|---------|------|------|--------|------|------|--------|------|--------|--------|--------|
|         | DIM  | TCGM | M3TFR2 | ITDM | TCGM | M3TFR2 | ITDM | TCGM   | M3TFR2 | ITDM   |
| $x_0^1$ | 300  | 60   | 119    | 37   | 178  | 370    | 140  | 0.0117 | 0.0246 | 0.0096 |
|         | 500  | 60   | 112    | 37   | 178  | 357    | 139  | 0.0178 | 0.0360 | 0.0145 |
|         | 1000 | 60   | 108    | 36   | 178  | 364    | 134  | 0.0330 | 0.0680 | 0.0259 |
|         | 2000 | 60   | 100    | 33   | 178  | 369    | 122  | 0.0632 | 0.1323 | 0.0459 |
| $x_0^2$ | 300  | 60   | 119    | 37   | 178  | 370    | 140  | 0.0119 | 0.0250 | 0.0097 |
|         | 500  | 60   | 112    | 37   | 178  | 357    | 139  | 0.0178 | 0.0361 | 0.0145 |
|         | 1000 | 60   | 108    | 36   | 178  | 364    | 134  | 0.0329 | 0.0679 | 0.0258 |
|         | 2000 | 60   | 100    | 33   | 178  | 369    | 122  | 0.0631 | 0.1319 | 0.0453 |
| $x_0^3$ | 300  | 43   | 76     | 29   | 127  | 227    | 106  | 0.0085 | 0.0152 | 0.0074 |
|         | 500  | 43   | 74     | 28   | 127  | 221    | 105  | 0.0127 | 0.0223 | 0.0109 |
|         | 1000 | 43   | 72     | 29   | 127  | 215    | 106  | 0.0234 | 0.0401 | 0.0205 |
|         | 2000 | 43   | 70     | 28   | 127  | 209    | 103  | 0.0450 | 0.0748 | 0.0381 |
| $x_0^4$ | 300  | 43   | 76     | 29   | 127  | 227    | 106  | 0.0084 | 0.0151 | 0.0073 |
|         | 500  | 43   | 74     | 28   | 127  | 221    | 105  | 0.0128 | 0.0224 | 0.0110 |
|         | 1000 | 43   | 72     | 29   | 127  | 215    | 106  | 0.0234 | 0.0402 | 0.0204 |
|         | 2000 | 43   | 70     | 28   | 127  | 209    | 103  | 0.0450 | 0.0748 | 0.0382 |

**Table 8.** Numerical results of Problem 8.

| $x_0$   | NI    |      |        |      | FE   |        |      | CPU    |        |        |
|---------|-------|------|--------|------|------|--------|------|--------|--------|--------|
|         | DIM   | TCGM | M3TFR2 | ITDM | TCGM | M3TFR2 | ITDM | TCGM   | M3TFR2 | ITDM   |
| $x_0^1$ | 3000  | 8    | 23     | 16   | 16   | 160    | 31   | 0.0011 | 0.0092 | 0.0034 |
|         | 5000  | 8    | 27     | 16   | 16   | 209    | 31   | 0.0015 | 0.0164 | 0.0050 |
|         | 10000 | 8    | 39     | 16   | 16   | 337    | 31   | 0.0028 | 0.0434 | 0.0092 |
|         | 20000 | 9    | 52     | 17   | 18   | 511    | 33   | 0.0053 | 0.1367 | 0.0179 |
| $x_0^2$ | 3000  | 15   | 26     | 15   | 57   | 180    | 57   | 0.0040 | 0.0129 | 0.0056 |
|         | 5000  | 15   | 31     | 16   | 57   | 234    | 61   | 0.0054 | 0.0229 | 0.0084 |
|         | 10000 | 15   | 43     | 16   | 57   | 362    | 61   | 0.0080 | 0.0532 | 0.0129 |
|         | 20000 | 15   | 58     | 16   | 57   | 553    | 61   | 0.0136 | 0.1349 | 0.0255 |
| $x_0^3$ | 3000  | 7    | 4      | 13   | 14   | 10     | 25   | 0.0012 | 0.0008 | 0.0034 |
|         | 5000  | 7    | 5      | 14   | 14   | 14     | 27   | 0.0015 | 0.0014 | 0.0048 |
|         | 10000 | 8    | 7      | 14   | 16   | 24     | 27   | 0.0028 | 0.0037 | 0.0083 |
|         | 20000 | 8    | 7      | 14   | 16   | 28     | 27   | 0.0048 | 0.0071 | 0.0147 |
| $x_0^4$ | 3000  | 12   | 10     | 13   | 45   | 29     | 49   | 0.0030 | 0.0022 | 0.0046 |
|         | 5000  | 12   | 9      | 13   | 45   | 27     | 49   | 0.0037 | 0.0025 | 0.0062 |
|         | 10000 | 13   | 10     | 14   | 49   | 34     | 53   | 0.0060 | 0.0047 | 0.0106 |
|         | 20000 | 13   | 12     | 14   | 49   | 44     | 53   | 0.0113 | 0.0109 | 0.0203 |

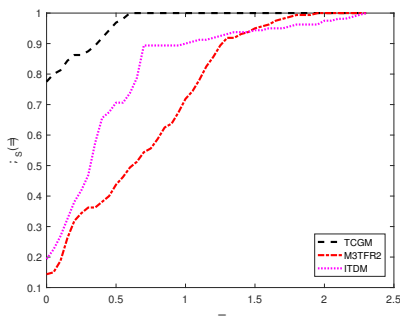
**Table 9.** Numerical results of Problem 9.

| $x_0$   | NI    |      |        |      | FE   |        |      | CPU    |        |        |
|---------|-------|------|--------|------|------|--------|------|--------|--------|--------|
|         | DIM   | TCGM | M3TFR2 | ITDM | TCGM | M3TFR2 | ITDM | TCGM   | M3TFR2 | ITDM   |
| $x_0^1$ | 3000  | 1    | 1      | 1    | 1    | 1      | 1    | 0.0002 | 0.0002 | 0.0002 |
|         | 5000  | 1    | 1      | 1    | 1    | 1      | 1    | 0.0003 | 0.0004 | 0.0003 |
|         | 10000 | 1    | 1      | 1    | 1    | 1      | 1    | 0.0007 | 0.0007 | 0.0007 |
|         | 20000 | 1    | 1      | 1    | 1    | 1      | 1    | 0.0013 | 0.0013 | 0.0013 |
| $x_0^2$ | 3000  | 19   | 53     | 19   | 93   | 447    | 111  | 0.0221 | 0.1023 | 0.0289 |
|         | 5000  | 19   | 65     | 23   | 93   | 586    | 136  | 0.0529 | 0.2467 | 0.0640 |
|         | 10000 | 19   | 81     | 23   | 93   | 839    | 138  | 0.0852 | 0.6982 | 0.1223 |
|         | 20000 | 19   | 109    | 24   | 93   | 1266   | 147  | 0.1505 | 2.1338 | 0.3055 |
| $x_0^3$ | 3000  | 21   | 31     | 19   | 106  | 187    | 110  | 0.0431 | 0.0489 | 0.0282 |
|         | 5000  | 21   | 35     | 19   | 106  | 243    | 110  | 0.0475 | 0.1227 | 0.0466 |
|         | 10000 | 21   | 46     | 21   | 105  | 368    | 124  | 0.1213 | 0.3983 | 0.1271 |
|         | 20000 | 21   | 60     | 19   | 107  | 544    | 111  | 0.2026 | 0.9904 | 0.2261 |
| $x_0^4$ | 3000  | 19   | 34     | 22   | 91   | 230    | 131  | 0.0412 | 0.0693 | 0.0356 |
|         | 5000  | 21   | 39     | 20   | 103  | 299    | 117  | 0.0614 | 0.1698 | 0.0552 |
|         | 10000 | 20   | 50     | 20   | 98   | 439    | 119  | 0.0856 | 0.3457 | 0.1516 |
|         | 20000 | 20   | 68     | 21   | 98   | 667    | 124  | 0.1671 | 1.2313 | 0.2964 |

**Table 10.** Numerical results of Problem 10.

| $x_0$   | NI    |      |        |      | FE   |        |      | CPU    |        |        |
|---------|-------|------|--------|------|------|--------|------|--------|--------|--------|
|         | DIM   | TCGM | M3TFR2 | ITDM | TCGM | M3TFR2 | ITDM | TCGM   | M3TFR2 | ITDM   |
| $x_0^1$ | 3000  | 47   | 55     | 46   | 156  | 231    | 240  | 0.0182 | 0.0293 | 0.0438 |
|         | 5000  | 47   | 60     | 49   | 156  | 282    | 270  | 0.0390 | 0.0554 | 0.0783 |
|         | 10000 | 47   | 68     | 47   | 158  | 371    | 203  | 0.0638 | 0.1485 | 0.1183 |
|         | 20000 | 47   | 80     | 53   | 158  | 515    | 287  | 0.1047 | 0.4317 | 0.2471 |
| $x_0^2$ | 3000  | 50   | 66     | 56   | 163  | 324    | 266  | 0.0226 | 0.0383 | 0.0470 |
|         | 5000  | 50   | 75     | 56   | 165  | 412    | 312  | 0.0382 | 0.0929 | 0.1007 |
|         | 10000 | 50   | 86     | 53   | 165  | 558    | 228  | 0.0632 | 0.2602 | 0.1246 |
|         | 20000 | 51   | 105    | 54   | 170  | 799    | 282  | 0.1357 | 0.6699 | 0.3113 |
| $x_0^3$ | 3000  | 44   | 43     | 54   | 142  | 129    | 278  | 0.0179 | 0.0255 | 0.0422 |
|         | 5000  | 44   | 43     | 52   | 142  | 130    | 255  | 0.0252 | 0.0241 | 0.0527 |
|         | 10000 | 44   | 44     | 45   | 142  | 137    | 274  | 0.0804 | 0.0732 | 0.1359 |
|         | 20000 | 44   | 47     | 50   | 144  | 149    | 244  | 0.0962 | 0.1259 | 0.2088 |
| $x_0^4$ | 3000  | 46   | 46     | 47   | 150  | 150    | 210  | 0.0237 | 0.0181 | 0.0287 |
|         | 5000  | 46   | 49     | 47   | 150  | 168    | 220  | 0.0347 | 0.0328 | 0.0489 |
|         | 10000 | 47   | 52     | 50   | 155  | 192    | 315  | 0.0682 | 0.0813 | 0.1844 |
|         | 20000 | 46   | 56     | 50   | 152  | 233    | 228  | 0.1068 | 0.1990 | 0.2056 |

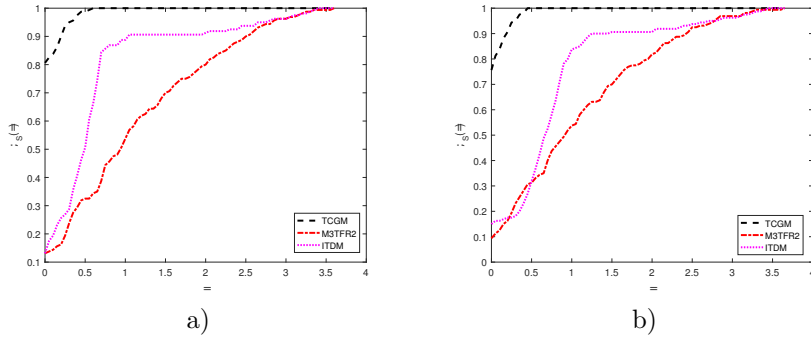
Dolan and Moré [6] performance profiles procedure is a tool for evaluating and comparing the performances of iterative methods. The profile of each method is measured according to the ratio of its computational outcome compared to the computational outcome of the best presented method. Figures 1–2 clearly show that *TCGM* method is the most efficient as compared to the other two methods.



**Figure 1.** Iterations performance profile

## 5 Conclusions

In this paper, we proposed a three-term conjugate gradient-based method (*TCGM*) for solving systems of large-scale nonlinear monotone equations. The proposed method is free from derivative evaluations and also satisfies the sufficient descent condition independent of any line search. Its global convergence was also established. The proposed algorithm was tested on some benchmark problems with different starting points and different dimensions and the numerical results show that the method is very efficient.



**Figure 2.** Results of numerical experiments: a) function evaluations performance profile  
b) CPU time performance profile

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