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ASYMPTOTIC METHOD FOR APPROXIMATION OF RESONANT INTERACTION OF NONLINEAR MULTIDIMENSIONAL HYPERBOLIC WAVES

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Abstract. A method of averaging along characteristics of weakly nonlinear hyperbolic systems, which was presented in earlier works of the author for one dimensional waves, is generalized for some cases of multidimensional wave problems. In this work we consider such systems and discuss a way to use the internal averaging along characteristics for new problems of asymptotical integration.

Key words: perturbation methods, averaging, nonlinear hyperbolic equations, waves' interaction, resonances.

1 State of Multidimensional Problems

Many physical systems, a state of which depend on the time, can be represented by the following system of equations

$$\frac{\partial}{\partial t}\begin{pmatrix}u_1\\u_2\\\vdots\\u_n\end{pmatrix}+A(U)\frac{\partial}{\partial x}\begin{pmatrix}u_1\\u_2\\\vdots\\u_n\end{pmatrix}+B(U)\frac{\partial}{\partial y}\begin{pmatrix}u_1\\u_2\\\vdots\\u_n\end{pmatrix}+C(U)\frac{\partial}{\partial z}\begin{pmatrix}u_1\\u_2\\\vdots\\u_n\end{pmatrix}=\varepsilon^2F\left[U\right], \tag{1.1}$$

where $U(x,y,z,t;\varepsilon)=(u_1,u_2,\cdots,u_n)$, $A(U)$, $B(U)$, $C(U)$ are $n\times n$ matrices, F is a nonlinear operator, ε is a small parameter. As examples of such systems we mention dispersive waves in plasma, problems of nonlinear optics, hydrodynamics equations, waves in elastic materials. For example, hydrodynamics equations [14]

$$\begin{cases} \dfrac{\partial \vec{v}}{\partial t} + \vec{v}\nabla\vec{v} + \dfrac{1}{\rho}\nabla p = 0, \quad p = P(\rho), \\ \dfrac{\partial \rho}{\partial t} + \nabla\left(\rho\,\vec{v}\right) = 0 \end{cases} \tag{1.2}$$

can be rewritten in form (1.1) with $U = (v_1, v_2, v_3, \rho)^T$ and with the following matrices $A(U)$, $B(U)$, $C(U)$:

$$\begin{pmatrix} v_1 & 0 & 0 & \frac{P'_\rho}{\rho} \\ 0 & v_1 & 0 & 0 \\ 0 & 0 & v_1 & 0 \\ \rho & 0 & 0 & v_1 \end{pmatrix}, \quad \begin{pmatrix} v_2 & 0 & 0 & 0 \\ 0 & v_2 & 0 & \frac{P'_\rho}{\rho} \\ 0 & 0 & v_2 & 0 \\ 0 & \rho & 0 & v_2 \end{pmatrix}, \quad \begin{pmatrix} v_3 & 0 & 0 & 0 \\ 0 & v_3 & 0 & 0 \\ 0 & 0 & v_3 & \frac{P'_\rho}{\rho} \\ 0 & 0 & \rho & v_3 \end{pmatrix}.$$

It easy to see that there is a constant solution

$$\vec{v}(x, y, z, t) = \vec{v}_0, \quad \rho(x, y, z, t) = \rho_0$$

of system (1.2). Small perturbations of the constant solution are typical objects of acoustics

$$\rho'(x, y, z, t) \ll \rho_0, \quad \vec{v}'(x, y, z, t) \ll 1.$$

The order of the value $\dfrac{\rho'}{\rho_0} = O(\varepsilon)$ is called the Mach number. Then the solution of problem (1.2) can be represented in the following form

$$\rho(x, y, z, t) = \rho_0 + \varepsilon\rho'(x, y, z, t), \quad \vec{v}(x, y, z, t) = \vec{v}_0 + \vec{v}'(x, y, z, t).$$

Note that the Mach number (and analogous known Reynolds, Prandtl, Rossby and other numbers) is only one possible source of small (or large) parameters in mathematical models.

There is a different way to introduce a small parameter into (1.1) model and then to use special methods of perturbation theory. An asymptotical approximation of solution of system (1.1) can be constructing in the following form

$$U(x, y, z, t; \varepsilon) = U_0 + \varepsilon U_1(x, y, z, t; \varepsilon),$$

where $U_1(t, x, y, z; \varepsilon)$ is an unknown function, which has a fixed asymptotical anzats. As an example of such anzats we mention Gardner–Morikawa transform (see [17]) $U_1(\varepsilon^{a+1}t, \varepsilon^a(x - \lambda_0 t))$, which uses ideas of two asymptotical scales and a method of characteristics. In the common case λ_0 is a constant, which must be find using standard perturbation technique. In this paper λ_0 is an eigenvalue of the non-perturbed problem.

In our algorithm we also use two general principles of asymptotical analysis. First, we introduce three slow variables

$$\tau = \varepsilon t, \quad \eta = \varepsilon y, \quad \zeta = \varepsilon z$$

and represent a solution of problem (1.1) as a linear combination of two functions depending on slow variables τ, η, ζ and on fast variables t, x, y, z:

$$U(x, y, z, t; \varepsilon) = U_0(\eta, \zeta, \tau) + \varepsilon U_1(x, \eta, \zeta, t; \varepsilon). \tag{1.3}$$

Remark 1. Similar ideas of asymptotical representation of unknown functions are used in many papers for description of different wave processes, and frequently the anzats of the solution is fixed even more strongly. For example, in [2, 18] for (1.1) type problems the anzats of the asymptotical solution has (in our notation) slow variables τ, η, ζ and fast variables t, x. However, in these papers the main terms of the asymptotical expression have only one harmonic $e^{i\omega t}$. In particular, such a strongly fixed anzats cannot describe an interaction of few harmonics of the periodical solutions. A more general anzats (1.3) gives a possibility to simulate the resonant interaction of periodical waves.

Substituting expression (1.3) into (1.1) and setting the coefficients at ε^n equal to zero we obtain for $n = 1$:

$$U_{0\tau} + B(U_0)U_{0\eta} + C(U_0)U_{0\zeta} = 0 \tag{1.4}$$

and for $n = 2$:

$$\begin{aligned} U_{1t} + A_0 U_{1x} = \varepsilon\big(&A_1(U_1)U_{1x} + B_0 U_{1\eta} + B_1(U_1)U_{0\eta} + C_0 U_{1\zeta} \\ &+ C_1(U_1)U_{0\zeta} + F[U_0]\big), \end{aligned} \tag{1.5}$$

where

$$A_0 = A(U_0),\ \ B_0 = B(U_0),\ \ C_0 = C(U_0),\ \ A_1(U_1) = ||a_{ij}||_{n\times n}\,,$$

$$a_{ij}(u_1, u_2, \dots, u_n) = \frac{\partial a_{ij}(U_0)}{\partial u_1}u_1 + \frac{\partial a_{ij}(U_0)}{\partial u_2}u_2 + \dots + \frac{\partial a_{ij}(U_0)}{\partial u_n}u_n.$$

Matrices $B_1 = ||b_{ij}||_{n\times n}$ and $C_1 = ||c_{ij}||_{n\times n}$ can be written analogically.

Let us assume that a unique solution of problem (1.4) exists in the region of slow variables (see, e.g., [1])

$$\Omega_{c_0} = \{(\tau, \eta, \zeta):\ 0 \leqslant \tau + |\eta| + |\zeta| \leqslant c_0\}\,.$$

Note that problem (1.4) has no asymptotical integration difficulties and it can be solved numerically by using standard approximations. Our goal is to construct a uniformly valid asymptotical solution of problem (1.5) in the large region of fast variables

$$\Omega_{c_0/\varepsilon} = \left\{(t, x, y, z):\ 0 \leqslant t + |x| + |y| + |z| \leqslant \frac{c_0}{\varepsilon}\right\}.$$

2 Method of Internal Averaging

Let us assume that non perturbed system (1.5) with $\varepsilon = 0$ is hyperbolic. Then it can be rewritten in the Riemann invariants (see, e.g., [15]). In this case there exists the non-degenerate matrix $R(\tau, \eta, \zeta)$, such that

$$\Lambda := R^{-1}A_0 R = \mathrm{d}iag\,(\lambda_1(\tau, \eta, \zeta), \lambda_2(\tau, \eta, \zeta), \dots, \lambda_n(\tau, \eta, \zeta))\,. \tag{2.1}$$

Thus using

$$R(t, \eta, \zeta)U_1(t, \eta, \zeta; \varepsilon) = U^1(t, \eta, \zeta; \varepsilon) = (u_1^1, u_2^1, \dots, u_n^1),$$

we obtain

$$\frac{\partial}{\partial t}U_1 = \varepsilon R_\tau^{-1}U + R^{-1}U_t, \;\; \frac{\partial}{\partial x}U_1 = R^{-1}U_x$$

and problem (1.5), (2.1) can be rewritten as:

$$\frac{\partial u_j}{\partial t} + \lambda_j(\tau,\eta,\zeta)\frac{\partial u_j}{\partial x} = \varepsilon f_j(\tau,\eta,\zeta,U,U_x,U_\eta,U_\zeta). \qquad (2.2)$$

For example, the system of hydrodynamics (1.2) is written in in the Riemann invariants as

$$u_1 = \frac{1}{2}\left(v_{10} + \frac{\sqrt{P'_\rho(\rho_0)}}{\rho_0}\rho\right), \quad u_2 = \frac{1}{2}\left(v_{10} - \frac{\sqrt{P'_\rho(\rho_0)}}{\rho_0}\rho\right),$$

$$u_3 = v_{30}, \quad u_4 = v_{40}.$$

For this case the coefficients λ_j in system (2.2) are given by:

$$\lambda_1 = v_{10} + \sqrt{P'_\rho(\rho_0)}, \;\; \lambda_2 = v_{10} - \sqrt{P'_\rho(\rho_0)}, \;\; \lambda_3 = \lambda_4 = v_{10}.$$

Functions f_j in system (2.2) are the following (here we omit terms, which define nonlinear operator $F[U]$ in (1.1)):

$$f_j = \sum_{i=1}^n\sum_{k=1}^n f_{jik}u_i\frac{\partial u_k}{\partial x} + \sum_{i=1}^n g_{ji}\frac{\partial u_i}{\partial \eta} + \sum_{i=1}^n h_{ji}\frac{\partial u_i}{\partial \zeta} + \sum_{i=1}^n p_{ji}u_i. \qquad (2.3)$$

When problem (1.4) is solved, then all coefficients f_{jik}, g_{ji}, h_{ji}, p_{ji} in (2.3) are known functions of variables τ, η, ζ. Therefore in order to get asymptotic approximation of (1.1), (1.3) we must solve system (2.2), (2.3).

Asymptotical integration of this system is a difficult problem even if λ_j in (2.2) are constants and it becomes even more complicated when λ_j are some functions. In order to construct an asymptotic, which is uniformly valid in region $\Omega_{\frac{c_0}{\varepsilon}}$, it is necessary to use special methods of asymptotical analysis. A survey of mathematical results for this problems is presented in [3].

A problem with periodical initial conditions deals with internal resonances and it is more difficult for analysis and numerical solution. A new method of asymptotical integration of equations (2.2) along characteristics was presented in [16] and developed in author's work [6] for one dimensional systems with internal resonances (without variables y and z and for constant coefficients λ_j).

In this paper, for two or three dimensional cases we propose a modified method of internal averaging along characteristic. Let

$$y_j = x - \frac{1}{\varepsilon}\int_0^\tau \lambda_j(s,\nu,\mu)\,ds$$

be fast characteristic variables. We will seek a solution of (2.2) in the following form

$$u_j(t,x;\eta,\zeta,\varepsilon) = v_j(\tau,y_j,\eta,\zeta) + o(1).$$

If $\lambda_j = const$ we obtain $y_j = x - \lambda_j t$ and functions v_j can be find by solving the following averaged system

$$\frac{\partial v_j}{\partial \tau} = M_j\left[f_j\right], \tag{2.4}$$

$$M_j\left[g(\tau,\dots,y_i,\dots,\eta,\zeta)\right] = \lim_{T\to+\infty}\frac{1}{T}\int_0^T g(\tau,\dots,y_j+(\lambda_j-\lambda_i)t,\dots,\eta,\zeta)\,dt. \tag{2.5}$$

Applications of this method for one dimensional problems with $\lambda_j = const$ are presented in our papers [5, 9, 10, 11, 12]. Note that these models are one dimensional simplifications of general models, that are two or three dimensional. For such models asymptotical analysis can be done by using the method developed in this paper.

The case $\lambda_j(\tau,\nu,\mu) \neq const$ is more difficult and it is not sufficiently explored. In this case the operator (2.5) of averaging along characteristics in system (2.4) must be changed by the operator

$$\lim_{\varepsilon\to0}\int_0^\tau g_j\big(s,\dots,y_j+\frac{1}{\varepsilon}\int_0^s\left(\lambda_j(r,\nu,\mu)-\lambda_i(r,\nu,\mu)\right)\,dr,\dots\big)\,ds. \tag{2.6}$$

It is easy to show, that formula (2.6) yields (2.5) if all $\lambda_j = const$. If (1.1) has no internal resonances then operator (2.6) can be rewritten as

$$\frac{1}{(2\pi)^{n-1}}\underbrace{\int_0^{2\pi}\cdots\int_0^{2\pi}}_{n-1} g(\tau,y_1,\dots,y_n)\,dy_1\dots dy_{j-1}dy_{j+1}\dots dy_n.$$

3 New Asymptotic Problems

Let be $\lambda_j \neq const$. From system (1.4) we see, that it can have a stationary solution $\lambda_j(\eta,\zeta)$. We can treat variables η, ζ as parameters and construct an asymptotical solution of the system

$$\frac{\partial u_j}{\partial t} + \lambda_j(\eta,\zeta)\frac{\partial u_j}{\partial x} = \varepsilon f_j\left(\cdots\right) \tag{3.1}$$

with periodical initial conditions

$$u_j(0,x,\eta,\zeta;\varepsilon) = u_{j0}(x,\eta,\zeta), \quad j=1,2,\dots,n. \tag{3.2}$$

Comparison with the case $\lambda_j = const$ shows, that we get a more difficult problem, which deals with internal resonances.

For solving problem (3.1), (3.2) ideas from our works [4, 7] can be applied. Let us assume that $\lambda_j(\tau) = \lambda_j(\tau; \eta_0, \zeta_0)$. Then we can construct the asymptotic near the fixed point (η_0, ζ_0) as a solution of the following system

$$\frac{\partial u_j}{\partial t} + \lambda_j(\tau)\frac{\partial u_j}{\partial x} = \varepsilon f_j(\cdots) \tag{3.3}$$

subject to initial conditions (3.2). Problem (3.2), (3.3) was considered in our work [13] with additional conditions for all coefficients $\lambda_j(\tau)$:

$$\frac{d}{d\tau}\left(\frac{\lambda_i(\tau) - \lambda_j(\tau)}{\lambda_k(\tau) - \lambda_j(\tau)}\right) \equiv 0. \tag{3.4}$$

From condition (3.4) it follows that there exist such functions $\alpha(\tau)$, $\beta(\tau)$ and constants λ_j^0, that

$$(\forall j)\ \lambda_j(\tau) = \lambda_j^0\alpha(\tau) + \beta(\tau).$$

This strong restriction is not satisfied for most real models. In our work [8] weaker conditions were formulated for system (3.3) with periodical initial conditions (3.2). In order to formulate these new restrictions the Wronskians

$$W^{(k)}_{(i_1,i_2,\ldots,i_r)}(\tau) \equiv \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \frac{d^k\lambda_{i_1}(\tau)}{d\tau^k} & \frac{d^k\lambda_{i_2}(\tau)}{d\tau^k} & \cdots & \frac{d^k\lambda_{i_r}(\tau)}{d\tau^k} \\ \frac{d^{r-1}\lambda_{i_1}(\tau)}{d\tau^{k+1}} & \frac{d^{k+1}\lambda_{i_2}(\tau)}{d\tau^{k+1}} & \cdots & \frac{d^{k+1}\lambda_{i_r}(\tau)}{d\tau^{k+1}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{d^{k+r-1}\lambda_{i_1}(\tau)}{d\tau^{k+r-1}} & \frac{d^{k+r-1}\lambda_{i_2}(\tau)}{d\tau^{k+r-1}} & \cdots & \frac{d^{k+r-1}\lambda_{i_r}(\tau)}{d\tau^{k+r-1}} \end{vmatrix}$$

and

$$W^{(0)}_{(i_1,i_2,\ldots,i_r)}(\tau) \equiv W_{(i_1,i_2,\ldots,i_r)}(\tau),\ W_{(1,2,\ldots,n)}(\tau) \equiv W(\tau)$$

are used. For our method $(\forall\tau \in [0, \tau_0])$ it is sufficient to require that

$$W(\tau) \neq 0$$

and it is necessary to fulfill conditions

$$W^{(k)}_{(i_1,i_2,\ldots,i_r)}(\tau) \neq 0 \text{ or } W^{(k)}_{(i_1,i_2,\ldots,i_r)}(\tau) \equiv 0$$

for $\forall(i_1, i_2, \ldots, i_r)\ 0 \leqslant i_1 < i_2 < \cdots < i_r \leqslant n\ \exists k \geqslant 0$. In this way we hope to apply the method of internal averaging along characteristic for real-world models and this topic is the object of our future research.

For system (3.1) and for system (3.3), (3.4) with fixed parameters η_0, ζ_0 we can apply our method [4, 7]. The idea of this method is to change coefficients by

$$\lambda_j(\eta_0, \zeta_0) = \alpha_j + \varepsilon\beta_j,$$

where α_j are selected in a special way. Let be

$$\delta_{j\vec{l}} = l_1(\alpha_1 - \alpha_j) + \cdots + l_{j-1}(\alpha_{j-1} - \alpha_j) + l_{j+1}(\alpha_{j+1} - \alpha_j) + \cdots + l_n(\alpha_1 - \alpha_n)$$

and a set of non-resonant vectors

$$R_j = \Big\{ \vec{l} = \big(l_1, \ldots, l_{j-1}, l_{j+1}, \ldots, l_n\big) \in R^{n-1} \setminus \Big\{\vec{0}\Big\} : \ \delta_{j\vec{l}} \neq 0 \Big\}$$

are given. Then coefficients α_j must satisfy the conditions

$$(\forall \vec{l} \in R_j) \quad \big\|\delta_{j\vec{l}}\big\| = \mathcal{O}\big(\|\vec{l}\|\big)^{-\gamma}, \quad \gamma > 0.$$

4 Conclusions

Principles of two asymptotical scales and averaging along characteristics are used in the paper for asymptotic analysis of multidimensional hyperbolic systems. These principles give asymptotical approximations for mathematical models of dispersive waves in plasma, problems of nonlinear optics, waves in elastic materials.

The main advantage of the presented method is that it gives a uniformly valid in a large region (of order $O(\varepsilon^{-1}$, where ε is a small parameter) asymptotic approximation of wave processes, which describe a resonant interaction of travelling periodical waves. In order to construct this approximation we need to solve integro-differential system of averaging equations. In particular cases the results of earlier works of the author for one dimensional waves can be easy modified for new multidimensional problems.

In the general case new problems of asymptotic integration must be solved. These problems are formulated in the paper.

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