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Cardinal Approximation of Functions by Splines on an Interval*

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Abstract. The cardinal interpolant of functions on the real line by splines is determined by certain formula free of solving large or infinite systems. We apply this formula to functions given on the interval $[0,1]$ introducing special extensions of functions from $[0,1]$ into the real line which maintains the optimal error estimates. The computation of the parameters determining the interpolant costs $O(n \log n)$ operations.

Key words: splines, interpolation, error estimates, best constants.

1 Introduction

Cardinal interpolation of functions on the real line by B-splines has good (in some sense best possible) approximation properties, and the parameters of the interpolant are determined by certain formulae free of solving large or infinite systems (see Section 2.2). In the present paper we apply this formula to functions determined on an interval, say $[0,1]$, introducing special extensions of functions given on $[0,1]$ onto the real line maintaining the error estimates. The computation of the parameters determining the interpolant costs $\mathcal{O}(n \log n)$ operations. This is somewhat more expensive than the interpolation through introducing suitable boundary conditions for the spline interpolant at points 0 and 1 and solving the corresponding band system that costs $\mathcal{O}(n)$ operations. On the other hand, the accuracy of the approximation and the computations are under a good control, and we are free of the care about the numerical stability of solving large systems.

In Sections 2 we recall some results about the cardinal interpolation of functions by smooth splines on the real line. In Section 3 we define sufficiently smooth extensions of functions f given on $[0,1]$ onto the real line that maintain the error estimates of the cardinal spline interpolation. The starting idea is to

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continue f with the help of the Taylor expansions at 0 and 1. Damping the Taylor polynomials we get more convenient schemes for the practice. Approximating the derivatives in the Taylor coefficients by suitable finite differences we obtain an interpolation projection of $f \in C[0,1]$ which asymptotically maintains the optimal accuracy of the basic cardinal interpolation on the real line. In Section 4 we call attention to another idea, when we decompose f into a periodic part and a polynomial.

2 Preliminaries

2.1 The father B-spline

For $m \in \mathbb{N}$, the father B-spline B_m of order m [1, 6] (or of degree $(m-1)$, as defined in [2, 7, 13]) can be defined by the formula

$$B_m(x) = \frac{1}{(m-1)!} \sum_{i=0}^{m} (-1)^i \binom{m}{i} (x-i)_+^{m-1}, \quad x \in \mathbb{R},$$

where, as usual, $0! = 1$, $0^0 := \lim_{x \downarrow 0} x^x = 1$,

$$\left(\begin{array}{c} m \\ i \end{array} \right) = \frac{m!}{i!\,(m-i)!}, \qquad (x-i)_+^{m-1} := \left\{ \begin{array}{cc} (x-i)^{m-1}, & x-i \geq 0 \\ 0, & x-i < 0 \end{array} \right\}.$$

Most important properties of B_m are as follows:

$$B_m \,|_{[i,i+1]} \in \mathcal{P}_{m-1},\ i \in \mathbb{Z}, \quad B_m \in C^{m-2}(\mathbb{R}),$$

$$\mathrm{supp} B_m = [0,m], \quad B_m(x) > 0 \quad \text{for } 0 < x < m, \quad \sum_{j \in \mathbb{Z}} B_m(x-j) = 1,\ x \in \mathbb{R},$$

where $\mathcal{P}_{m-1}$ is the space of polynomials of degree $\leq m-1$.

2.2 Cardinal interpolation by splines on the real line

Let us introduce the following spaces of functions defined on the real line $\mathbb{R} = (-\infty, \infty)$:

- $BC(\mathbb{R})$ is the Banach space of bounded continuous functions equipped with the norm
$$\|f\|_\infty = \|f\|_{\infty,\mathbb{R}} = \sup_{x \in \mathbb{R}} |f(x)|;$$

- $W^{m,\infty}(\mathbb{R})$ is the Sobolev space of functions having bounded derivatives up to order m (the derivatives are understood in the sense of distributions);

- $V^{m,\infty}(\mathbb{R})$ is the space of functions with bounded mth derivative; clearly, $W^{m,\infty}(\mathbb{R}) + \mathcal{P}_m \subset V^{m,\infty}(\mathbb{R})$ (this inclusion is strict);

- $V_h^{m,\infty}(\mathbb{R})$ is the space of functions $f \in C^{m-2}(\mathbb{R})$ such that

$$f^{(m-1)} \mid_{(ih,(i+1)h)} \in C((ih,(i+1)h)),$$

$$f^{(m)} \mid_{(ih,(i+1)h)} \in L^{\infty}((ih,(i+1)h)) \text{ for all } i \in \mathbb{Z},$$

$$\sigma_{h,m}(f) = \sigma_{h,m,\infty}(f) := \sup_{i \in \mathbb{Z}} \sup_{ih<x<(i+1)h} \mid f^{(m)}(x) \mid < \infty;$$

  clearly, $V^{m,\infty}(\mathbb{R}) \subset V_h^{m,\infty}(\mathbb{R})$ and $\|f^{(m)}\|_\infty = \sigma_{h,m}(f)$ for $f \in V^{m,\infty}(\mathbb{R})$;

- $S_{h,m}(\mathbb{R}) \subset V_h^{m,\infty}(\mathbb{R})$ is the space of cardinal splines of order m (or of degree $m-1$) with the knot set $h\mathbb{Z} = \{ih : i \in \mathbb{Z}\}$ of the step size $h > 0$; thus $f_h \in S_{h,m}(\mathbb{R})$ means that $f_h \in C^{m-2}(\mathbb{R})$ and its restriction to any interval $(ih,(i+1)h)$, $i \in \mathbb{Z}$, belongs to $\mathcal{P}_{m-1}$.

It is known [3, 4, 5] that for $f \in BC(\mathbb{R})$, there exists a unique bounded spline $Q_{h,m}f \in S_{h,m}(\mathbb{R})$ (the interpolant of f) such that

$$(Q_{h,m}f)\big((k+\frac{m}{2})h\big) = f\big((k+\frac{m}{2})h\big), \quad k \in \mathbb{Z}; \tag{2.1}$$

if $f \in C(\mathbb{R})$ is polynomially growing as $x \to \infty$ or $x \to -\infty$, the interpolant $Q_{h,m}f \in S_{h,m}(\mathbb{R})$ still exists and is unique in the space of polynomially growing functions. Cases $m = 1$ and $m = 2$ are trivial in the sense that the spline interpolant can be constructed on every subinterval $[ih,(i+1)h]$, $i \in \mathbb{Z}$, independently of other other subintervals. For $m \geq 3$, $Q_{h,m}f$ can be represented in the form [7]

$$(Q_{h,m}f)(x) = \sum_{k \in \mathbb{Z}} d_k B_m(h^{-1}x - k), \; x \in \mathbb{R},$$

$$d_k = \sum_{j \in \mathbb{Z}} \alpha_{k-j,m} f((j+\frac{m}{2})h), \; k \in \mathbb{Z}, \tag{2.2}$$

where B_m is the father B-spline introduced in Section 2.1,

$$\alpha_{k,m} = \sum_{l=1}^{m_0} \frac{z_{l,m}^{m_0-1}}{P_m'(z_{l,m})} z_{l,m}^{|k|}, \quad k \in \mathbb{Z}, \quad m_0 = \begin{cases} (m-2)/2, & m \text{ even}, \\ (m-1)/2, & m \text{ odd}; \end{cases} \tag{2.3}$$

$z_{l,m} \in (-1,0)$, $l = 1,...,m_0$, are the roots of the characteristic polynomial $P_m(z) = \sum_{|k| \leq m_0} B_m(k+\frac{m}{2})z^{k+m_0}$ (they are simple and $1/z_{l,m} \in (-\infty,-1)$, $l = 1,...,m_0$, are the remaining m_0 roots of $P_m \in \mathcal{P}_{2m_0}$). Clearly,

$$\mid \alpha_{k,m} \mid \leq c_m \theta_m^k, \quad \theta_m = \max_{1 \leq l \leq m} \mid z_{l,m} \mid < 1.$$

For $h = 1/n$, $n \in \mathbb{N}$, the complexity of $(Q_{h,m}f)(x)$ for x restricted to $[0,1]$ is $\mathcal{O}(n \log n)$ arithmetical operations: first, due to the support properties of B_m, the formula for $(Q_{h,m}f)(x)$ in (2.2) reduces for $x \in [ih,(i+1)h)$ to $(Q_{h,m}f)(x) = \sum_{k=i-m+1}^{i} d_k B_m(h^{-1}x - k)$, thus we have to determine $n + 2m$ parameters d_k; second, since $\alpha_{k,m}$ decays exponentially, one d_k can be computed by (2.2), (2.3) with a given accuracy $\mathcal{O}(h^r)$, $r > 0$, in $\mathcal{O}(\log n)$ arithmetical operations.

Theorem 1. *For $f \in V_h^{m,\infty}(\mathbb{R})$, the error $f - Q_{h,m}f$ is bounded in $\mathbb{R}$ and*

$$\|f - Q_{h,m}f\|_\infty \le \Phi_{m+1}\pi^{-m}h^m\sigma_{h,m}(f) \tag{2.4}$$

where $\Phi_m = \dfrac{4}{\pi}\sum_{k=0}^\infty \dfrac{(-1)^{km}}{(2k+1)^m}$ is the Favard constant.

In particular,

$$\Phi_1 = 1, \quad \Phi_2 = \pi/2, \quad \Phi_3 = \pi^2/8, \quad \Phi_4 = \pi^3/24, \quad \Phi_5 = 5\pi^4/384,$$

and it holds

$$\Phi_1 < \Phi_3 < \Phi_5 < \ldots < \frac{4}{\pi} < \ldots < \Phi_6 < \Phi_4 < \Phi_2, \quad \lim_{m\to\infty}\Phi_m = \frac{4}{\pi}.$$

The proof of Theorem 1 can be found in [2] for 1-periodic functions f and in [9] or [10] for the general formulation. In the periodic case with $h = 1/n$, even n, the estimate (2.4) realizes the best possible approximation of the periodic Sobolev class $W^{m,\infty}(\mathbb{R})$ in the sense of the Kolmogorov $n-$width, see [2]. Further, in the general (nonperiodic) formulation, estimate (2.4) is the best possible in the sense of the worst case for the classes $V_h^{m,\infty}(\mathbb{R})$ and $W^{m,\infty}(\mathbb{R})$ over all approximation methods using the same information as the cardinal spline interpolant – the values of f at the points $(k + \frac{m}{2})h)$, $k \in \mathbb{Z}$, see [9, 10, 12] for a more detailed formulation and a proof. In particular, the constant in (2.4) is exact (cannot be improved for the classes $V_h^{m,\infty}(\mathbb{R})$ and $W^{m,\infty}(\mathbb{R})$).

3 Taylor and Damped Taylor Extension of Functions from $[0,1]$ to $\mathbb{R}$

Now we assume that $h = 1/n$ and $f \in V_h^{m,\infty}[0,1]$, i.e., $f \in C^{m-2}[0,1]$,

$$f^{(m-1)}\,|_{(ih,(i+1)h)} \in C((ih,(i+1)h)), \quad f^{(m)}\,|_{(ih,(i+1)h)} \in L^\infty((ih,(i+1)h)),$$
$$i = 0,\ldots,n-1,$$

$$\sigma_{h,m,[0,1]}(f) := \sup_{0\le i\le n-1}\ \sup_{ih<x<(i+1)h} |\, f^{(m)}(x)\,| < \infty.$$

Extending f up to a function $\overline{f} \in V_h^{m,\infty}(\mathbb{R})$ defined on $\mathbb{R}$ such that

$$\sigma_{h,m,\mathbb{R}}(\overline{f}) = \sigma_{h,m,[0,1]}(f), \tag{3.1}$$

we obtain by Theorem 1 for the spline approximation $f_{h,m} := (Q_{h,m}\overline{f})|_{[0,1]}$ the error estimate

$$\|f - f_{h,m}\|_{\infty,[0,1]} := \sup_{0\le x\le 1}|f(x) - f_{h,m}(x)| \le \Phi_{m+1}\pi^{-m}h^m\sigma_{h,m,[0,1]}(f), \tag{3.2}$$

that is our final goal. Thus the problem reduces to the construction of an extension of a given function $f \in V_h^{m,\infty}[0,1]$ up to a function $\overline{f} \in V_h^{m,\infty}(\mathbb{R})$ such that (3.1) holds true. Below we discuss some ways how to do this.

3.1 Using the Taylor expansions

A simple extension of $f \in V_h^{m,\infty}[0,1]$ into $\overline{f} = f_T \in V_h^{m,\infty}(\mathbb{R})$ satisfying (3.1) can be constructed using the Taylor expansions:

$$f_T(x) = \begin{cases} \sum_{k=0}^{m-2} f^{(k)}(0)x^k/k!, & x < 0, \\ f(x), & 0 \le x \le 1, \\ \sum_{k=0}^{m-2} f^{(k)}(1)(x-1)^k/k!, & x > 1. \end{cases} \tag{3.3}$$

3.2 Damping the Taylor expansions

Although the Taylor extension f_T defined by (3.3) theoretically solves our problem in a simple way, it is not the simplest extension for the practice. Extensions $\overline{f}$ with supports in a small neighbourhood of $[0,1]$ are preferable when the parameters $d_k = \sum_{j\in\mathbb{Z}} \alpha_{k-j,m}\overline{f}((j+\frac{m}{2})h)$ of the interpolant $Q_{h,m}\overline{f}$ are computed, cf. (2.2). Below we present an extension $\overline{f}_T$ supported on $[-(m-1)h, 1+(m-1)h]$ and based on the damping the Taylor polynomials in (3.3). Namely, represent the Taylor polynomials in (3.3) in the basis $(x+jh)^{m-1}$, $j = 0,\dots,m-1$, of $\mathcal{P}_{m-1}$ for $x < 0$, and in the basis $(x-1-jh)^{m-1}$, $j = 0,\dots,m-1$, of $\mathcal{P}_{m-1}$ for $x > 1$:

$$\begin{aligned} \sum_{k=0}^{m-2} f^{(k)}(0)x^k/k! &= \sum_{j=0}^{m-1} \gamma_j (x+jh)^{m-1}, \\ \sum_{k=0}^{m-2} f^{(k)}(1)(x-1)^k/k! &= \sum_{j=0}^{m-1} \gamma'_j (x-1-jh)^{m-1}. \end{aligned} \tag{3.4}$$

Having determined the coefficients γ_j and γ'_j, put

$$\overline{f}_T(x) = \begin{cases} \sum_{j=0}^{m-1} \gamma_j (x+jh)_+^{m-1}, & x < 0, \\ f(x), & 0 \le x \le 1, \\ \sum_{j=0}^{m-1} \gamma'_j (x-1-jh)_-^{m-1}, & x > 1, \end{cases} \tag{3.5}$$

where

$$(x+jh)_+^{m-1} = \left\{ \begin{array}{ll} (x+jh)^{m-1}, & x+jh \ge 0 \\ 0, & x+jh < 0 \end{array} \right\},$$

$$(x-1-jh)_-^{m-1} = \left\{ \begin{array}{ll} (x-1-jh)^{m-1}, & x-1-jh < 0 \\ 0, & x-1-jh \ge 0 \end{array} \right\}$$

are functions belonging to $S_{h,m}(\mathbb{R})$. Then really $\overline{f}_T(x) = 0$ for $x < -(m-1)h$ and for $x > 1+(m-1)h$. Further,

$$f_T(x) - \overline{f}_T(x) = \begin{cases} \sum_{j=0}^{m-1} \gamma_j (x+jh)_-^{m-1}, & x < 0, \\ 0, & 0 \le x \le 1, \\ \sum_{j=0}^{m-1} \gamma'_j (x-1-jh)_+^{m-1}, & x > 1, \end{cases}$$

and we observe that $f_T - \overline{f}_T \in S_{h,m}(\mathbb{R}) \subset V_h^{m,\infty}(\mathbb{R})$, $\sigma_{h,m,\infty}(f_T - \overline{f}_T) = 0$. Hence $\overline{f}_T = f_T - (f_T - \overline{f}_T) \in V_h^{m,\infty}(\mathbb{R})$ and (3.1) holds true for $\overline{f}_T$:

$$\sigma_{h,m,\infty}(\overline{f}_T) = \sigma_{h,m,\infty}\big(f_T - (f_T - \overline{f}_T)\big) = \sigma_{h,m,\infty}(f_T) = \sigma_{h,m,[0,1]}(f).$$

To compute $Q_{h,m}\overline{f}_T$ by (2.2), we have to determine the missing values of $\overline{f}_T$ at the interpolation points belonging to the intervals $(-(m-1)h, 0)$ and $(1, 1+(m-1)h))$. For even m, we need the values $\overline{f}_T(-ih)$ and $\overline{f}_T(1+ih)$, $i = 1, ..., m-2$, and they are given by

$$\overline{f}_T(-ih) = \sum_{j=i+1}^{m-1} (h^{m-1}\gamma_j)(j-i)^{m-1}, \quad \overline{f}_T(1+ih) = \sum_{j=i+1}^{m-1} (h^{m-1}\gamma_j')(i-j)^{m-1}. \tag{3.6}$$

For odd m, we need the values $\overline{f}_T(-(i-\frac{1}{2})h)$ and $\overline{f}_T(1+(i-\frac{1}{2})h)$, $i = 1, \ldots, m-1$, and they are given by

$$\overline{f}_T(-(i-\frac{1}{2})h) = \sum_{j=i}^{m-1} (h^{m-1}\gamma_j)(j-i+\frac{1}{2})^{m-1},$$

$$\overline{f}_T(1+(i-\frac{1}{2})h) = \sum_{j=i}^{m-1} (h^{m-1}\gamma_j')(i-j-\frac{1}{2})^{m-1}, \quad i = 1, \ldots, m-1. \tag{3.7}$$

These formulae involve coefficients $h^{m-1}\gamma_j$ and $h^{m-1}\gamma_j'$ for $j = 1, \ldots, m-1$; the coefficients $h^{m-1}\gamma_0$ and $h^{m-1}\gamma_0'$ (for even m also $h^{m-1}\gamma_1$ and $h^{m-1}\gamma_1'$) are redundant.

Let us comment on the computation of $h^{m-1}\gamma_j$, $j = 1, \ldots, m-1$. Differentiating equality (3.4) i times and setting $x = 0$ we obtain after elementary manipulations the system of linear algebraic equations with respect to $h^{m-1}\gamma_j$, $j = 1, \ldots, m-1$:

$$\sum_{j=1}^{m-1} j^{m-1-i}(h^{m-1}\gamma_j) = \frac{(m-1-i)!}{(m-1)!} h^i f^{(i)}(0), \quad i = 0, ..., m-2. \tag{3.8}$$

If we want to compute also $h^{m-1}\gamma_0$, differentiate (3.4) once more obtaining

$$\sum_{j=0}^{m-1} (h^{m-1}\gamma_j) = 0. \tag{3.9}$$

By the way, system (3.8), (3.9) becomes a Vandermonde system with respect to $h^{m-1}\gamma_j$, $j = 0, ..., m-1$. In Example 3, we exploit this fact.

Similarly, with respect to $h^{m-1}\gamma_j'$, $j = 1, ..., m-1$, we obtain the system

$$\sum_{j=1}^{m-1} j^{m-1-i}(h^{m-1}\gamma_j') = \frac{(-1)^{m-1-i}(m-1-i)!}{(m-1)!} h^i f^{(i)}(1), \quad i = 0, ..., m-2. \tag{3.10}$$

In the practice, it is sufficient to solve only one of systems (3.8) and (3.10) since $\overline{f}_T(1+ih)$ and $\overline{f}_T(1+(i-\frac{1}{2})h)$ can be determined respectively from $\overline{f}_T(-ih)$ and $\overline{f}_T(-(i-\frac{1}{2})h)$ and vice versa also introducing in (3.5) the change variables $x' = 1-x$.

Example 1. For $m=3$ (quadratic splines), system (3.8) has the form

$$(h^2\gamma_1) + 4(h^2\gamma_2) = f(0), \quad (h^2\gamma_1) + 2(h^2\gamma_2) = \frac{1}{2}hf'(0),$$

and its solution is

$$h^2\gamma_1 = -f(0) + hf'(0), \quad h^2\gamma_2 = \frac{1}{2}f(0) - \frac{1}{2}hf'(0).$$

Now (3.7) yields

$$\overline{f}_T(-\frac{1}{2}h) = \frac{7}{8}f(0) - \frac{5}{16}hf'(0), \qquad \overline{f}_T(-\frac{3}{2}h) = \frac{1}{8}f(0) - \frac{1}{16}hf'(0),$$
$$\overline{f}_T(1+\frac{1}{2}h) = \frac{7}{8}f(1) + \frac{5}{16}hf'(1), \qquad \overline{f}_T(1+\frac{3}{2}h) = \frac{1}{8}f(1) + \frac{1}{16}hf'(1).$$

Example 2. For $m=4$ (cubic splines), system (3.8) reads as

$$(h^3\gamma_1) + 8(h^3\gamma_2) + 27(h^3\gamma_3) = f(0),$$
$$(h^3\gamma_1) + 4(h^3\gamma_2) + 9(h^3\gamma_3) = \frac{1}{3}hf'(0),$$
$$(h^3\gamma_1) + 2(h^3\gamma_2) + 3(h^3\gamma_3) = \frac{1}{6}h^2f''(0).$$

Eliminating $(h^3\gamma_1)$ we find that

$$h^3\gamma_2 = -\frac{1}{2}f(0) + \frac{2}{3}hf'(0) - \frac{1}{4}h^2f''(0),$$
$$h^3\gamma_3 = \frac{1}{6}f(0) - \frac{1}{6}hf'(0) + \frac{1}{18}h^2f''(0),$$

and (3.6) yields

$$\overline{f}_T(-h) = \frac{5}{6}f(0) - \frac{2}{3}hf'(0) + \frac{7}{36}h^2f''(0),$$
$$\overline{f}_T(-2h) = \frac{1}{6}f(0) - \frac{1}{6}hf'(0) + \frac{1}{18}h^2f''(0);$$

by symmetry,

$$\overline{f}_T(1+h) = \frac{5}{6}f(1) + \frac{2}{3}hf'(1) + \frac{7}{36}h^2f''(1),$$
$$\overline{f}_T(1+2h) = \frac{1}{6}f(1) + \frac{1}{6}hf'(1) + \frac{1}{18}h^2f''(1).$$

Example 3. For $m \geq 3$, let $f \in V_h^{m,\infty}[0,1]$ satisfy the boundary conditions $f^{(i)}(0) = f^{(i)}(1) = 0$, $i = 1, ..., m-2$, which appear after a suitable change of variables, see [8] or [11]. System (3.9), (3.8), with the converse ordering of equations in (3.8), reads as

$$\sum_{j=0}^{m-1} j^k (h^{m-1}\gamma_j) = \delta_{k,m-1} f(0), \quad k = 0, ..., m-1,$$

where $\delta_{k,m-1}$ is the Kronecker symbol. By the Cramer rule,

$$h^{m-1}\gamma_j = D_j/D, \quad j = 1, ..., m-1,$$

where

$$D = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & 1^1 & 2^1 & \dots & (m-2)^1 & (m-1)^1 \\ 0 & 1^2 & 2^2 & \dots & (m-2)^2 & (m-1)^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1^{m-1} & 2^{m-1} & \dots & (m-2)^{m-1} & (m-1)^{m-1} \end{vmatrix} = 1!\,2!...(m-1)!$$

is the determinant of the system (it is the Vandermonde determinant determined by the numbers 0, 1, …, $m-1$), and D_j is obtained from D replacing the column $(1, j. ..., j^{m-1})^{\mathrm{T}}$ by the column of free terms:

$$D_j = \begin{vmatrix} 1 & 1 & \dots & 1 & 0 & 1 & \dots & 1 \\ 0 & 1^1 & \dots & (j-1)^1 & 0 & (j+1)^1 & \dots & (m-1)^1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1^{m-2} & \dots & (j-1)^{m-2} & 0 & (j+1)^{m-2} & \dots & (m-1)^{m-2} \\ 0 & 1^{m-1} & \dots & (j-1)^{m-1} & f(0) & (j+1)^{m-1} & \dots & (m-1)^{m-1} \end{vmatrix}$$

$$= (-1)^{m+j+1} f(0) \begin{vmatrix} 1 & \dots & 1 & 1 & \dots & 1 \\ 0 & \dots & (j-1)^1 & (j+1)^1 & \dots & (m-1)^1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & (j-1)^{m-2} & (j+1)^{m-2} & \dots & (m-1)^{m-2} \end{vmatrix}$$

$$= (-1)^{m-j-1} \frac{1!\,2! \dots (m-1!)}{j!(m-1-j)!} f(0)$$

(the last determinant is again Vandermonde). Here we have exploited the well known formula for a Vandermonde determinant:

$$\begin{vmatrix} 1 & 1 & ... & 1 \\ z_0 & z_1 & ... & z_m \\ z_0^2 & z_1^2 & ... & z_m^2 \\ \vdots & \vdots & \vdots & \vdots \\ z_0^m & z_1^m & ... & z_m^m \end{vmatrix} = \prod_{0 \leq l < k \leq m} (z_k - z_l).$$

Thus

$$h^{m-1}\gamma_j = (-1)^{m-j-1} \frac{1}{j!(m-1-j)!}, \quad j = 1, ..., m-1,$$

and (3.6)/(3.7) takes the following form: for even m,

$$\overline{f}_T(-ih) = \beta_{i,m} f(0), \quad \overline{f}_T(1+ih) = \beta_{i,m} f(1),$$
$$\beta_{i,m} = \sum_{j=i+1}^{m-1} \frac{(-1)^{m-1-j}}{j!(m-1-j)!}(j-i)^{m-1}, \quad i = 1, ..., m-2;$$

for odd m,

$$\overline{f}_T(-(i-\frac{1}{2})h)) = \beta_{i,m} f(0), \quad \overline{f}_T(1+(i-1/2)h)) = \beta_{i,m} f(1),$$
$$\beta_{i,m} = \sum_{j=i}^{m-1} \frac{(-1)^{m-1-j}}{j!(m-1-j)!}(j-i+1/2)^{m-1}, \quad i = 1, ..., m-1.$$

3.3 Approximating $f^{(i)}(0)$ and $f^{(i)}(1)$ by finite differences

It follows from the structure of systems (3.8) and (3.10) that

$$\max_{1\le j\le m-1} h^{m-1}|\gamma_j| \le c \max_{0\le i\le m-2} h^i|f^{(i)}(0)|,$$
$$\max_{1\le j\le m-1} h^{m-1}|\gamma'_j|) \le c \max_{0\le i\le m-2} h^i|f^{(i)}(1)|,$$

where the constant c depends only on m. Assume now that $f \in C^m[0,1]$. Approximate the derivatives $f^{(i)}(0)$ and $f^{(i)}(1)$ by the finite differences based on the differentiation of the Lagrange interpolation polynomials of degree m with the interpolation points $0, h, ..., mh$ and $1-mh, ..., 1-h, 1$, respectively, in case of even m, and $h/2, 3h/2, ..., (2m-1)h/2$ and $1-(2m-1)h/2, ...1-h/2$, respectively, in case of odd m. The error of this approximation is of order $o(h^{m-i})$ that can be seen with the help of Banach–Steinhaus theorem. Thus $h^i f^{(i)}(0)$ and $h^i f^{(i)}(1)$ are approximated with the accuracy $o(h^m)$. Hence also the error of the solution $h^{m-1}\gamma_j$, $j = 1, ..., m-1$, of system (3.8) and the error of the solution $h^{m-1}\gamma'_j$, $j = 1, ..., m-1$, of the solution of system (3.10) are of order $o(h^m)$, and the same is true for the accuracy of the values of $\overline{f}_T$ defined in (3.6)/(3.7). In its turn, this implies that the spline approximation $f_{h,m} = (Q_{h,m}\overline{f}_T)|_{[0,1]}$ is reproduced with the accuracy $o(h^m)$ in the uniform norm $\| \cdot \|_{\infty,[0,1]}$: denoting the constructed spline approximation of f by $P_{h,m}f$, we have

$$\| f_{h,m} - P_{h,m}f \|_{\infty,[0,1]} \le \varepsilon_{h,f} h^m \| f^{(m)} \|_{\infty,[0,1]},$$

where $\varepsilon_{h,f} \to 0$ as $n \to \infty$ (as $h \to 0$) for any fixed $f \in C^m[0,1]$. Unfortunately, the best constant $\Phi_{m+1}\pi^{-m}$ in (3.2) is not maintained for the error of $P_{h,m}f$ but, nevertheless, accuracy (3.2) is achieved for any fixed $f \in C^m[0,1]$ asymptotically as $n \to \infty$:

$$\| f - P_{h,m}f \|_{\infty,[0,1]} \le c_m h^m \| f^{(m)} \|_{\infty,[0,1]}, \tag{3.11}$$
$$\| f - P_{h,m}f \|_{\infty,[0,1]} \le (\Phi_{m+1}\pi^{-m} + \varepsilon_{h,f}) h^m \| f^{(m)} \|_{\infty,[0,1]} . \tag{3.12}$$

Note that (3.11) remains to be true even if we use polynomials of degree $m-1$ when the derivatives $f^{(i)}(0)$ and $f^{(i)}(1)$ are approximated by finite differences; degree m is necessary for the validity of (3.12).

The interpolant $P_{h,m}f$ is well defined for any $f \in C[0,1]$. For even m, $P_{h,m}f$ is uniquely determined by the grid values $f(jh)$, $j = 0,1,...,n$; for odd m, $P_{h,m}f$ is uniquely determined by the grid values $f((j-\frac{1}{2})h)$, $j = 1,...,n$. The assignment $f \longmapsto P_{h,m}f$ defines a projector in $C[0,1]$, i.e., $P_{h,m}^2 = P_{h,m}$, although $f \longmapsto (Q_{h,m}\overline{f}_T)|_{[0,1]}$ is not a projection. A modified approximation $P_{h,m}f$ is useful if f is given and C^m-smooth in a neighbourhood of $[0,1]$ – we may use central (symmetric) differences for the approximation of $f^{(i)}(0)$ and $f^{(i)}(1)$ that reduces the values of c_m and $\varepsilon_{h,f}$ in (3.11) and (3.12).

4 Extracting a Periodic Part and a Polynomial Part of f

Denote by Z_i the space of polynomials z of degree not exceeding $i+1$ and satisfying $\int_0^1 z(x)\,dx = 0$. There exist unique polynomials $z_i \in Z_i$, $i = 0,1,...,$ such that

$$z_i^{(k)}(1) - z_i^{(k)}(0) = \delta_{i,k} \quad \text{(Kronecker symbol)}, \quad i,k = 0,1,2,\ldots . \tag{4.1}$$

Indeed, fixing i, we have to determine the coefficients of $z_i(x) = \sum_{j=0}^{i+1} c_{ij}x^j$ so that (4.1) holds and $\int_0^1 z_i(x)\,dx = 0$. For $k \geq i+1$, (4.1) is trivially fulfilled, so c_{ij}, $j = 0,...,i+1$, must be determined so that (4.1) is satisfied for $k = 0,...,i$ and $\int_0^1 z_i(x)\,dx = 0$. These conditions yield an $i+2$ dimensional triangular system with nonzeroes on the main diagonal uniquely determining c_{ij}, $j = 0,...,i+1$. Clearly, $z_0(x) = x - \frac{1}{2}$. Having z_i in hand, it is easy to check that z_{i+1} satisfies the recursion

$$z_{i+1}(x) = \int_0^x z_i(x)\,dx + \int_0^1 xz_i(x)\,dx, \quad i = 0,1,\ldots .$$

The first polynomials z_i are as follows:

$$z_0(x) = x - \frac{1}{2}, \quad z_1(x) = \frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{12}, \quad z_2(x) = \frac{1}{6}x^3 - \frac{1}{4}x^2 + \frac{1}{12}x,$$
$$z_3(x) = \frac{1}{24}x^4 - \frac{1}{12}x^3 + \frac{1}{24}x^2 - \frac{1}{720}, \ldots .$$

Denote by $\widetilde{C}^l[0,1]$ the subspace of functions $\widetilde{f} \in C^l[0,1]$ satisfying the periodic boundary conditions $\widetilde{f}^{(i)}(1) - \widetilde{f}^{(i)}(0) = 0$, $i = 0,\ldots,l$. By

$$\widetilde{P}_l f = f - \sum_{i=0}^{l}[f^{(i)}(1) - f^{(i)}(0)]z_i$$

is defined a projector $\widetilde{P}_l$ in $C^l[0,1]$ that projects $C^l[0,1]$ onto $\widetilde{C}^l[0,1]$ and induces the direct sum

$$C^l[0,1] = \widetilde{P}_l C^l[0,1] \dotplus (I - \widetilde{P}_l)C^l[0,1],$$
$$\widetilde{P}_l C^l[0,1] = \widetilde{C}^l[0,1], \quad (I - \widetilde{P}_l)C^l[0,1] = Z_l.$$

Thus every function $f \in C^l[0,1]$ has a unique representation

$$f = \widetilde{f} + p, \quad p = \sum_{i=0}^{l} [f^{(i)}(1) - f^{(i)}(0)] z_i \in Z_l, \quad \widetilde{f} = f - p \in \widetilde{C}^l[0,1]. \quad (4.2)$$

An 1-periodic extension of a function $\widetilde{f} \in \widetilde{C}^l[0,1]$ is $C^l(\mathbb{R})$-smooth and we maintain for the extension the notation $\widetilde{f}$. Respectively, (4.2) defines an extension of $f \in C^l[0,1]$ into the sum $\overline{f} = \widetilde{f} + p \in C^l(\mathbb{R})$ of an 1-periodic function $\widetilde{f} \in \widetilde{C}^l[0,1]$ and a polynomial p of degree $\le l+1$.

Using this extension for $f \in V_h^{m,\infty}[0,1] \subset C^{m-2}[0,1]$ we have $l = m-2$,

$$\overline{f} = \widetilde{f} + p, \quad p = \sum_{i=0}^{m-2} [f^{(i)}(1) - f^{(i)}(0)] z_i \in Z_{m-2} \subset \mathcal{P}_{m-1} \subset S_{h,m},$$

hence $Q_{h,m} p = p$,

$$Q_{h,m}\overline{f} = Q_{h,m}\widetilde{f} + p, \quad \sigma_{h,m,\infty}(\overline{f}) = \sigma_{h,m,\infty}(\widetilde{f}) = \sigma_{h,m,[0,1]}(f),$$

and error estimate (2.4) yields

$$\| f - Q_{h,m}\overline{f} \|_{\infty,[0,1]} \le \| \overline{f} - Q_{h,m}\overline{f} \|_{\infty,\mathbb{R}} \le \Phi_{m+1} \pi^{-m} h^m \sigma_{h,m,[0,1]}(f).$$

Thus the approximation of a function $f \in V_h^{m,\infty}[0,1]$ by splines is reduced to the interpolation of 1-periodic function $\widetilde{f}$, and the optimal accuracy is maintained. For a periodic function the construction of the spline interpolant is well elaborated, see, e.g., [11]. Unfortunately, the approach of the present section is restricted to the case where we really know the values of $f^{(i)}(1) - f^{(i)}(0)$, $i = 0, ..., m-2$, since approximating the derivatives by finite differences as in Section 3.3, only the accuracy $O(h^2)$ is achieved.

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