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Extrapolation of Tikhonov Regularization Method*

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Abstract. We consider regularization of linear ill-posed problem $Au = f$ with noisy data f_δ, $\|f_\delta - f\| \le \delta$. The approximate solution is computed as the extrapolated Tikhonov approximation, which is a linear combination of $n \ge 2$ Tikhonov approximations with different parameters. If the solution u_* belongs to $\mathcal{R}((A^*A)^n)$, then the maximal guaranteed accuracy of Tikhonov approximation is $O(\delta^{2/3})$ versus accuracy $O(\delta^{2n/(2n+1)})$ of corresponding extrapolated approximation. We propose several rules for choice of the regularization parameter, some of these are also good in case of moderate over- and underestimation of the noise level. Numerical examples are given.

Keywords: ill-posed problems, regularization, Tikhonov method, extrapolation, noise level, regularization parameter choice, balancing principle, monotone error rule, rule R2.

AMS Subject Classification: 65J20; 47A52.

1 Introduction

Extrapolation as a tool for increasing the accuracy of approximation methods is based on the idea to form the approximate solution as a linear combination of $n \ge 2$ approximations with different values of parameter, where the coefficients are chosen in such a way that the leading terms in error expansion will be eliminated. Extrapolation is widely used in discretization methods, in numerical integration, in interpolation etc [16, 21].

In this paper we consider an operator equation

$$Au = f, \quad f \in \mathcal{R}(A),$$

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where $A \in L(H, F)$ is a linear continuous operator between Hilbert spaces H and F. We suppose that instead of exact data $f \in F$ noisy data $f_\delta \in F$ with $\|f_\delta - f\| \le \delta$ is available. To approximate the solution $u_* \in H$ of this equation, we use the Tikhonov method $u_\alpha = (\alpha I + A^*A)^{-1}A^*f_\delta$, where $\alpha > 0$ and I is the identity operator. In case $H = F$, $A = A^* \ge 0$ also the Lavrentiev method $u_\alpha = (\alpha I + A)^{-1}f_\delta$ may be used. In this paper we consider extrapolated version of the Tikhonov method.

Up to now, to the extrapolation for increasing the accuracy of approximation methods in ill-posed problems only few papers are devoted. In [21] (see also [34]) the extrapolated Tikhonov method and a version of the extrapolated Lavrentiev method were proposed for systems of linear algebraic equations. The extrapolated Tikhonov method for operator equations with exact data was studied in [10]. In [5, 6, 33] some other techniques for extrapolation of Tikhonov method for ill-conditioned linear systems were proposed. In case of noisy data the extrapolated Tikhonov method was studied in [11, 12].

Another, more known possibility for increasing the accuracy of regularization methods is the use of iteration. Then we get the m-iterated Tikhonov approximation $u_\alpha = u_{m,\alpha}$ by starting with $u_{0,\alpha} = 0$ and iteratively computing the approximations

$$u_{i,\alpha} = (\alpha I + A^*A)^{-1}(\alpha u_{i-1,\alpha} + A^*f_\delta) \quad (i = 1, \ldots, m). \tag{1.1}$$

Using different parameters α_i at different iteration steps i gives nonstationary iterated Tikhonov approximation [14].

$$u_{i,\alpha_i} = (\alpha_i I + A^*A)^{-1}(\alpha u_{i-1,\alpha_{i-1}} + A^*f_\delta) \quad (i = 1, \ldots, m) \tag{1.2}$$

As shown in [11], this approximation coincides with extrapolated approximation v_{n,α_k} for $k = 1$, $n = m$:

$$v_{n,\alpha_k} = \sum_{i=k}^{n+k-1} d_i u_{\alpha_i}, \qquad d_i = \prod_{j=k, j\ne i}^{n+k-1} \Big(1 - \frac{\alpha_i}{\alpha_j}\Big)^{-1}. \tag{1.3}$$

It is easy to see that $\sum_{i=k}^{n+k-1} d_i = 1$ but the sequence of coefficients d_i has alternating signs, hence v_{n,α_k} is not a convex combination of terms u_{α_i}. Note that for large n and close α_i's (1.2) is a more stable way to find v_{m,α_i} than (1.3). In case of smooth solution

$$u_* \in \mathcal{R}((A^*A)^{p/2}) \tag{1.4}$$

a proper choice of α gives for extrapolated Tikhonov approximation $u_{\text{appr}} = v_{n,\alpha_k}$ the error estimate

$$\|u_{\text{appr}} - u_*\| \le \text{const}\, \delta^{p/(p+1)} \tag{1.5}$$

with $p \le 2n$. Note that in a posteriori choice of the regularization parameter in m-iterated Tikhonov method approximations are often computed for some sequence $\{\alpha_i\}$ of parameters, until some condition is fulfilled, and a single approximation with maximal accuracy $\mathcal{O}(\delta^{\frac{2m}{2m+1}})$ is used. One example is the balancing principle, advocated recently in many papers [1, 2, 3, 4, 8, 17, 18, 19, 20, 22, 23, 24, 25, 28]. The accuracy of the Tikhonov approximation ($m = 1$) is low but increasing the number m of iterations also increases the amount of computational work, since at transition from u_{m,α_i} to $u_{m,\alpha_{i+1}}$ we have to solve m equations. The approximation computed by extrapolated Tikhonov method has both benefits: its accuracy is the same as in $n = m$ times iterated Tikhonov method but at the transition from v_{m,α_i} to $v_{m,\alpha_{i+1}}$ only one equation needs to be solved.

In this paper we use the same approach as in papers [11, 12] and propose various new rules for a posteriori choice of the regularization parameter. In case of moderate overestimation of the noise level our rules R2e and Me give essentially better results than other rules. In contrast to other rules, rules R2 and R2e also allow moderate underestimation of the noise level.

The plan of this paper is as follows. Sections 2, 3 are devoted to parameter choices in (iterated) Tikhonov method and in (extrapolated) Tikhonov method, respectively. Many order optimal parameter choice rules in Tikhonov method require computing $u_{2,\alpha}$ and/or $u_{3,\alpha}$. In Section 3 we formulate analogous rules for (extrapolated) Tikhonov method, using the extrapolated Tikhonov approximations instead of the iterated Tikhonov approximations. In Section 4 we formulate the monotone error rule for choosing an approximation from sequence. In the last section results of numerical experiments for proposed rules are given.

2 Parameter Choice in the (Iterated) Tikhonov Method

For a posteriori choice of the regularization parameter α in (iterated) Tikhonov method several rules are proposed. In discrepancy principle [26, 36] for (iterated) Tikhonov approximation $u_{m,\alpha}$ the parameter α is chosen in such a way that $\|Au_{m,\alpha} - f_\delta\| = b\delta$, $b \geq 1$. In the modified discrepancy principle [9, 29] α_{MD}, in the monotone error rule [35] (ME-rule) α_{ME} and in rule R2 [31] α_{R2} are chosen from equations

$$(Au_{m,\alpha} - f_\delta, Au_{m+1,\alpha} - f_\delta) = b\delta, \qquad b \geq 1, \tag{2.1}$$

$$\frac{(Au_{m,\alpha} - f_\delta, Au_{m+1,\alpha} - f_\delta)}{\|Au_{m+1,\alpha} - f_\delta\|} = b\delta, \qquad b \geq 1, \tag{2.2}$$

$$\frac{\kappa(\alpha)\|A^*(Au_{m+1,\alpha} - f)\|^2}{\sqrt{\alpha}(A^*(Au_{m+1,\alpha}-f), A^*(Au_{m+2,\alpha}-f))^{1/2}} = b\delta, \;\; \kappa(\alpha) = 1 + \frac{\alpha}{\|A\|^2}, \tag{2.3}$$

respectively. Due to the equality

$$A^*(Au_{m+1,\alpha} - f) = \alpha(u_{m,\alpha} - u_{m+1,\alpha})$$

the equation (2.3) may be written as

$$d_{\mathrm{R2}}(\alpha) := \frac{\sqrt{\alpha}\|u_{m,\alpha} - u_{m+1,\alpha}\|^2\kappa(\alpha)}{(u_{m,\alpha} - u_{m+1,\alpha}, u_{m+1,\alpha} - u_{m+2,\alpha})^{1/2}} = b\delta. \tag{2.4}$$

In case (1.4) the discrepancy principle, modified discrepancy principle, ME-rule and R2-rule (under certain assumption, see [31]) guarantee the error estimate (1.5) for $p \le 1$, $p \le 2$, $p \le 2$ and $p \le 2$, respectively. The name of the ME-rule is justified by the property

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\|u_\alpha - u_*\| \ge 0 \qquad \text{for all } \alpha \in (\alpha_{\mathrm{ME}}, \infty).$$

It means that the optimal parameter

$$\alpha_{\mathrm{opt}} = \operatorname{argmin}\{\|u_\alpha - u_*\|, \alpha > 0\} \le \alpha_{\mathrm{ME}}.$$

In numerical experiments of Section 5 we get good results with estimated smaller parameter $\alpha_{\mathrm{MEe}} = \min(0.5\alpha_{\mathrm{ME}},\ 0.6\alpha_{\mathrm{ME}}^{1.08})$.

The equation (2.4) may have many solutions and both the smallest and the largest solutions may be of interest. Here we propose to take the largest solution as α_{R2}. Then typically $\alpha_{\mathrm{R2}} \ge \alpha_{\mathrm{opt}}$ and the estimated smaller parameter $\alpha_{\mathrm{R2e}} := 0.5\alpha_{\mathrm{R2}}$ is typically better than α_{R2}. Extensive numerical experiments have shown that in case $\|f - f_0\| = \delta$ typically α_{MEe} is the better of the parameters α_{MEe} and α_{R2e}, whereas in case $\|f - f_0\| < \delta$, α_{R2e} is better but in both cases

$$\alpha_{\mathrm{Me}} := \min\{\alpha_{\mathrm{MEe}}, \alpha_{\mathrm{R2e}}\} \tag{2.5}$$

chooses the best of them.

Recently many papers [1, 2, 3, 4, 8, 17, 18, 19, 20, 22, 23, 24, 25, 28] advocate the balancing principle (also called Lepskii principle). Here the approximations u_{α_i} are computed for values $\alpha_1 = \delta^2$ and $\alpha_k = \alpha_1 q^{1-k}$, $k = 2, 3, \dots, M$, where $q < 1$ and M is such that $\alpha_{M-1} < 1 \le \alpha_M$. The regularization parameter is chosen as α_m, where m is the first index, for which a certain condition is fulfilled. For Tikhonov method this condition is in [23, 24]

$$\|u_{\alpha_{m+1}} - u_{\alpha_m}\| > \frac{c\delta}{\sqrt{\alpha_m}} \tag{2.6}$$

with $c = 2$ and in [17, 28]

$$\exists j \in \{1, \dots, m\}:\ \|u_{\alpha_{m+1}} - u_{\alpha_j}\| > \frac{c\delta}{\sqrt{\alpha_j}} \tag{2.7}$$

with $c = 2$. However, a proper c must depend on q in such a way that $c \to 0$ as $q \to 1$. Otherwise, after finding $\alpha_m \approx \alpha_{\mathrm{opt}}$ on coarse mesh and then refining the mesh, the left hand side of (2.6) tends to zero as $q \to 1$, hence α_m chosen by (2.6) increases. More precisely, it is shown in [13] that if in conditions (2.6), (2.7) the constant c satisfies $c > q^{-1} - 1$ and $c > q^{-1} - q^{m-j}$, respectively, then the error of Tikhonov approximation is a monotonically increasing function of c. As proven in [13, 30], the balancing principle with condition (2.6) is order optimal, if $c \ge 3\sqrt{3}(1-q)/(16\sqrt{q})$. We recommend to use the last constant in condition (2.6) and $c = (1-q^{m+1-j})/q$ in condition (2.7). Condition (2.7) needs huge computation time and resulted in large error in numerical experiments, therefore we used the condition

$$\exists j \in \{m-1, m\}:\ \frac{4\|u_{\alpha_{m+1}} - u_{\alpha_j}\|}{(1-q^{m+1-j})q^{(j-m-1)/2}} > \frac{\delta}{\sqrt{\alpha_j}} \tag{2.8}$$

in Section 5 besides of (2.7). In case of correlated noise we recommend to substitute in (2.8) the constant 4 by 3.

Consider a posteriori choice of the regularization parameter α in (extrapolated) Tikhonov method. In rules (2.1) and (2.2) of Section 2 the iterated approximation $u_{2,\alpha}$ is used for parameter choice in Tikhonov method, hence one additional equation must be solved. However, order optimal error estimates for source-like solutions remain true, if in these rules $u_{2,\alpha}$ is replaced by a proper linear combination of two approximations (see also [32]). We formulate corresponding rules for Tikhonov method and for extrapolated Tikhonov method. Consider set $\alpha_i = q^i$, $q < 1$, $i = 1, 2, \ldots$ We choose in discrepancy principle and in rules MD, ME and R2 parameter α_{MD}, α_{ME} and α_{R2} in (extrapolated) Tikhonov approximation v_{n,α_i} as $\alpha = \alpha_i$ fulfilling the conditions

$$\|Av_{n,\alpha_i} - f_\delta\| \approx b\delta, \qquad b \geq 1,$$

$$(Av_{n,\alpha_i} - f_\delta, Av_{n+1,\alpha_i} - f_\delta) \approx b\delta, \qquad b \geq 1,$$

$$d_{\mathrm{ME}}(\alpha_i) = \frac{(Av_{n,\alpha_i} - f_\delta, Av_{n+1,\alpha_i} - f_\delta)}{\|Av_{n+1,\alpha_i} - f_\delta\|} \approx b\delta, \qquad b \geq 1, \tag{2.9}$$

$$d_{\mathrm{R2}}(\alpha_i) = \frac{\sqrt{\alpha_i}\|v_{n,\alpha_i} - v_{n+1,\alpha_i}\|^2 \kappa(\alpha_i)}{(v_{n,\alpha_i} - v_{n+1,\alpha_i}, v_{n+1,\alpha_i} - v_{n+2,\alpha_{i-1}})^{1/2}} \approx b\delta, \tag{2.10}$$

respectively. We use the same formulas for parameter estimation as in (iterated) Tikhonov method:

$$\alpha_{\mathrm{MEe}} = \min(0.5\alpha_{\mathrm{ME}}, 0.6\alpha_{\mathrm{ME}}^{1.08}),\ \alpha_{\mathrm{R2e}} = 0.5\alpha_{\mathrm{R2}},\ \alpha_{\mathrm{Me}} = \min\{\alpha_{\mathrm{MEe}}, \alpha_{\mathrm{R2e}}\}.$$

3 Choice of Parameters in the Extrapolated Tikhonov Method

In $v_{n,\alpha}$ both indices can be viewed as regularization parameters. In the following we consider separately the cases, when one of parameters n and α is fixed and the other parameter is the regularization parameter.

1) Let the sequence $\alpha_k \geq \alpha_{k+1} \geq \ldots$ be given (α_k is fixed) and consider choice of n in extrapolated Tikhonov approximation v_{n,α_k}. We give condition for checking, whether v_{n,α_k} is a more accurate solution than v_{n-1,α_k}. Denote $r_n \equiv Av_{n,\alpha_k} - f_\delta$. Let $C = \mathrm{const} > 1$.

Theorem 1. [11]. *The functions* $d_D(n) = \|r_n\|$, $d_{ME}(n) = \dfrac{(r_n + r_{n+1}, r_{n+1})}{2\|r_{n+1}\|}$ *are monotonically decreasing and* $d_D(n+1) < d_{ME}(n) < d_D(n)$ *for all* n. *Let* n_D, n_{ME} *be the first numbers with* $d_D(n) \leq C\delta$, $d_{ME}(n) \leq C\delta$ *respectively. Then* $n_D - 1 \leq n_{ME} \leq n_D$ *and*

$$\|v_{n,\alpha_k} - u_*\| < \|v_{n-1,\alpha_k} - u_*\|, \quad n = 1, 2, \ldots, n_{ME}.$$

If a monotonically decreasing infinite sequence $\alpha_1, \alpha_2, \ldots$ *satisfies conditions*

$$\sum_{i=k}^{\infty} \alpha_i^{-1} = \infty, \quad \alpha_n^{-1} \leq const \sum_{i=k}^{n-1} \alpha_i^{-1},$$

then the existence of finite n_D and n_{ME} is guaranteed, for $n \in \{n_D, n_{ME}\}$, $\|v_{n,\alpha_k} - u_*\| \to 0$ ($\delta \to 0$), and in case (1.4) *the error estimate* (1.5) *holds for all $p > 0$.*

2) Let $n \geq 2$ and sequence $\alpha_1 > \alpha_2 > \ldots$ be fixed. Consider choice of $\alpha = \alpha_i$ in extrapolated approximation v_{n,α_i}.

Theorem 2. [11]. *The functions $d_D(\alpha) = \|Av_{n,\alpha} - f_\delta\|$, $d_{MD}(\alpha) = (Av_{n,\alpha} - f_\delta, Av_{n+1,\alpha} - f_\delta)$ are monotonically decreasing. If α is chosen from the discrepancy principle $d_D(\alpha) \approx C\delta$, then $\|v_{n,\alpha} - u_*\| \to 0$ ($\delta \to 0$) and in case* (1.4) *for $u_{appr} = v_{n,\alpha}$ the error estimate* (1.5) *holds in extrapolated Tikhonov method with $p \leq 2n - 1$. If α in extrapolated Tikhonov method is chosen from the modified discrepancy principle $d_{MD}(\alpha) \approx C\delta$, then $\|v_{n,\alpha} - u_*\| \to 0$ ($\delta \to 0$) and in case* (1.4) *for $u_{appr} = v_{n,\alpha}$ the error estimate* (1.5) *holds with $p \leq 2n$.*

In extrapolated Tikhonov approximation for the choice of the parameter α_i we also use ME-rule and R2-rule with formulas (2.9), (2.10) and corresponding estimates α_{MEe}, α_{R2e} and α_{Me}.

Consider now extrapolation of m times iterated method of Tikhonov (1.1). For different $\alpha_i = q_i\alpha$ ($i = 1, \ldots, n$) different number of iterations $m_1, \ldots, m_n$ may be used. As approximate solution we take (see [11])

$$v_{n,\alpha} = \sum_{i=k}^{n+k-1} \sum_{j=1}^{m_i} d_{i,j} u_{j,\alpha_i}, \tag{3.1}$$

where the coefficients $d_{i,j}$ can be uniquely determined from relation

$$\sum_{i=k}^{n+k-1} \sum_{j=1}^{m_i} d_{i,j}(1 + \lambda/q_i)^{-j} = \prod_{i=1}^{n}(1 + \lambda/q_i)^{-m_i} \quad (\forall \lambda \in \mathbb{R}).$$

Theorem 3. [11]. *If n and $q_1, \ldots, q_n$ are fixed and α is chosen from the discrepancy principle $d_D(n) \approx C\delta$, then $\|v_{n,\alpha} - u_*\| \to 0$ ($\delta \to 0$) and in case* (1.4) *the error estimate* (1.5) *holds for $u_{appr} = v_{n,\alpha}$ in* (3.1) *with $p \leq 2(m_1 + m_2 + \cdots + m_n) - 1$.*

4 The Monotone Error Rule for Choosing an Approximation from the Sequence

In balancing principle a sequence of approximate solutions $\{u_{\alpha_i}\}$ is computed and a rule for choice of one approximation u_{α_i} is given. It motivates us to give another rule, the monotone error rule, for choice of a proper approximation from the sequence.

Theorem 4. *Let $u_i = A^* w_i$, $i = 1, 2, \ldots$ be a sequence of approximations to solution u_* of the equation $Au = f$. Let i_{ME} be the first index i satisfying*

$$d_{ME}(i) = \frac{(Au_i - f_\delta + Au_{i+1} - f_\delta, w_{i+1} - w_i)}{2\|w_{i+1} - w_i\|} \leq \delta.$$

Then

$$\|u_i - u_*\| \leq \|u_{i-1} - u_*\| \qquad \text{for all } i = 2, \ldots, i_{ME}.$$

Proof. We have

$$\begin{aligned}\|u_i - u_*\|^2 - \|u_{i-1} - u_*\|^2 &= (u_{i-1} + u_i - 2u_*, u_i - u_{i-1}) \\ &= (Au_{i-1} + Au_i - 2f, w_i - w_{i-1}) \\ &= (Au_{i-1} - f_\delta + Au_i - f_\delta + 2(f_\delta - f),\, w_i - w_{i-1}) \\ &\le 2\|w_i - w_{i-1}\|[\delta - d_{\mathrm{ME}}(i-1)].\end{aligned}$$

Therefore, if $d_{\mathrm{ME}}(i-1) > \delta$, then $\|u_i - u_*\| < \|u_{i-1} - u_*\|$. □

To use the functional $d_{\mathrm{ME}}(i)$, elements w_i are needed. They may be found by computing at first w_i and on final step $u_i = A^* w_i$. The last theorem may be applied to many kinds of approximations: to approximations $u_i = u_{\alpha_i}$ with decreasing parameters $\alpha_1 > \alpha_2 > \dots$ in Tikhonov method or in iterated Tikhonov method. In extrapolated Tikhonov method i in u_i may refer to number of terms n in linear combination (1.3) or to α_i in (1.3) or to some other element in arbitrary sequence of extrapolated approximations.

5 Numerical Experiments

We solved 10 test problems from [15] *baart*, *deriv2*, *foxgood*, *gravity*, *heat*, *i_ laplace*, *phillips*, *shaw*, *spikes*, *wing* and another 6 from [7]: *gauss, hilbert, lotkin, moler, pascal, prolate*. We used discretization parameter $N = 100$ and if the problem had more parameters, then all of these had the default value 1. As in paper [7] we combined all 16 discretized problems with 6 solution vectors $\overline{u}_i = 1$, $\overline{u}_i = i/N$, $\overline{u}_i = ((i - \lfloor N/2 \rfloor)/\lceil N/2 \rceil)^2$, $\overline{u}_i = \sin 2\pi(i-1)/N$, $\overline{u}_i = i/N + 1/4 \sin 2\pi(i-1)/N$ and $\overline{u}_i = 0$ for $i \le \lfloor N/2 \rfloor$, 1 for $i > \lfloor N/2 \rfloor$. In 10 problems from [15] the original solution was also included into the test set. Besides solutions u_* we solved variants of the problems with smoothened solutions $(A^*A)^{p/2} u_*$, where $p = 0.25$, 0.5, 0.75, 1, 1.5, 2, 4, 8; the right-hand side was computed as $f = A(A^*A)^{p/2} u_*$. All problems were normalized in such a way that the norms of operator and right-hand side were equal to 1.

Instead of exact data f it is assumed that randomly perturbed data f_δ is given. The problems were regularized by the Tikhonov method, where Tikhonov approximations were computed on the set of alpha-values $\Omega = \{\alpha_i\}$ with $\alpha_i = q^i$, using $q = 0.9$. To choose $\alpha_i \in \Omega$ in the Tikhonov method, the following rules were applied.

1) Discrepancy principle: α_{D} is the first α_i, for which $\|Au_{\alpha_i} - f_\delta\| \le \delta$.

2) Monotone error rule: α_{ME} is the first α_i satisfying $d_{\mathrm{ME}}(\alpha_i) \le \delta$; $\alpha_{\mathrm{MEe}} = \min(0.5\alpha_{\mathrm{ME}}, 0.6\alpha_{\mathrm{ME}}^{1.08})$.

3) Rule R2: α_{R2} is the first α_i satisfying $d_{\mathrm{R2}}(\alpha_i) \le c_k \delta$, $c_1 = 0.3$, $c_2 = 0.2$, $c_3 = 0.13$; $\alpha_{\mathrm{R2e}} = 0.5\alpha_{\mathrm{R2}}$.

4) Balancing principle: here, in opposite to other rules, an increasing sequence α_1, α_2, …, $\alpha_1 = \delta^2$ and $\alpha_k = \alpha_1 q^{1-k}$, $k = 2, 3, \dots$ is used; α_{B1} and $\alpha_{\mathrm{B1*}}$ were chosen as the first α_m, for which (2.6) holds with $c = 2$ and $c = 3\sqrt{3}(1-q)/(16\sqrt{q})$, respectively; α_{B2} and $\alpha_{\mathrm{B2*}}$ were chosen as the first

α_m, for which (2.7) holds with $c_m = 2$ and $c_m = (1 - q^{m+1-j})/q$, respectively; α_{B3} was chosen as the first α_m, for which (2.8) holds.

We computed the extrapolated approximations

$$v_{n,\alpha_k} = \sum_{i=k}^{n+k-1} d_i u_{\alpha_i}, \quad d_i = \prod_{j=k, j\neq i}^{n+k-1} \frac{\alpha_j}{\alpha_j - \alpha_i} = \prod_{j=k, j\neq i}^{n+k-1} \frac{1}{1 - q^{i-j}}. \tag{5.1}$$

In this approximation in addition to rules MEe, R2e, Me we computed the estimated parameter α_{De}, using parameters α_{D}, $\alpha_{2\mathrm{D}}$, $\alpha_{3\mathrm{D}}$ from discrepancy principle with $n = 1, 2, 3$, respectively, and α was chosen as the nearest $\alpha \in \Omega$ to the alpha-value $\alpha_{n\mathrm{D}}^{c_{n,1}} \alpha_{\mathrm{D}}^{c_{n,2}}$, where $(c_{21}, c_{22}) = (1.22, -0.12)$, $(c_{31}, c_{32}) = (1.16, -0.04)$. These constants and constants in α_{MEe}, α_{Me}, α_{R2e} were found by optimization on a large data set. We use notations 2MEe, 2Me, 3De for rules MEe, Me, De, applied to extrapolated approximations with $n = 2$, $n = 2$, $n = 3$, respectively.

Table 1. Means of error ratios.

p	e_{D}	$e_{\mathrm{D'}}$	e_{MEe}	$e_{\mathrm{MEe'}}$	e_{R2e}	$e_{\mathrm{R2e'}}$	e_{Me}	$e_{\mathrm{Me'}}$	e_{MEs}	$e_{\mathrm{MEs'}}$
0	1.26	2.21	1.19	1.90	1.42	1.61	1.19	1.60	1.20	1.94
0.25	1.46	3.28	1.39	2.82	1.90	2.31	1.38	2.21	1.39	2.89
0.5	1.56	4.20	1.51	3.60	2.09	2.57	1.49	2.55	1.50	3.71
0.75	1.59	4.93	1.54	4.35	2.17	3.01	1.53	2.84	1.53	4.50
1	1.82	4.98	1.55	4.82	2.31	2.88	1.53	2.87	1.52	5.00
1.5	2.47	3.95	1.26	3.80	1.40	1.67	1.24	1.67	1.25	3.98
2	2.73	3.86	1.21	3.39	1.15	1.31	1.18	1.31	1.19	3.57
4	2.92	3.85	1.20	3.33	1.13	1.25	1.18	1.25	1.18	3.50
8	2.95	3.85	1.20	3.33	1.13	1.24	1.18	1.24	1.18	3.50
mean	2.08	3.90	1.34	3.48	1.63	1.98	1.32	1.95	1.33	3.62

Table 2. Means of error ratios for balancing principle.

p	e_{B1}	$e_{\mathrm{B1'}}$	$e_{\mathrm{B1*}}$	$e_{\mathrm{B1*'}}$	e_{B2}	$e_{\mathrm{B2'}}$	$e_{\mathrm{B2*}}$	$e_{\mathrm{B2*'}}$	e_{B3}	$e_{\mathrm{B3'}}$
0	6.21	7.60	2.29	2.86	3.77	4.57	2.67	3.30	1.77	2.14
0.25	11.97	15.32	3.59	4.70	6.65	8.37	4.33	5.61	2.51	3.26
0.5	20.47	27.09	4.92	6.78	10.41	13.71	6.12	8.32	3.22	4.38
0.75	30.02	40.21	6.29	8.84	14.50	19.70	8.00	11.16	3.97	5.46
1	36.63	50.86	7.19	10.3	17.71	24.22	9.20	13.34	4.53	6.23
1.5	38.99	56.33	6.18	9.20	17.49	24.96	8.17	12.11	3.98	5.23
2	38.07	55.35	5.73	8.66	16.91	24.35	7.65	11.50	3.83	4.77
4	38.46	55.81	5.66	8.55	17.07	24.52	7.54	11.40	3.87	4.68
8	38.72	55.80	5.63	8.52	17.13	24.53	7.50	11.37	3.86	4.65
mean	28.84	40.48	5.27	7.60	13.52	18.77	6.80	9.79	3.50	4.53

Rule MEs chooses the parameter $\alpha = \alpha_k$ in approximation (5.1) as the nearest $\alpha \in \Omega$ to $\min\{0.5\alpha_i, 0.6\alpha_i^{1.08}\}$, where α_i is found by applying Theorem 4 to the sequence $u_i = v_{n,\alpha_i}$. As opposed to previous rules, where the regularization parameter was α, we computed approximations v_{maxD} as (5.1), choosing the regularization parameter n by Theorem 1.

Table 3. Means of error ratios for extrapolated approximations, $n = 2$.

p	e_{De}	$e_{\mathrm{De'}}$	e_{MEe}	$e_{\mathrm{MEe'}}$	e_{R2e}	$e_{\mathrm{R2e'}}$	e_{Me}	$e_{\mathrm{Me'}}$	e_{MEs}	$e_{\mathrm{MEs'}}$
0	1.20	2.07	1.19	1.83	1.48	1.52	1.31	1.51	1.19	1.85
0.25	1.41	3.17	1.39	2.68	1.99	2.16	1.57	2.05	1.39	2.71
0.5	1.51	4.17	1.49	3.30	2.15	2.34	1.66	2.31	1.49	3.34
0.75	1.46	5.05	1.48	3.75	2.47	2.85	1.61	2.53	1.47	3.79
1	1.37	5.63	1.43	3.96	2.26	3.11	1.52	2.62	1.42	4.00
1.5	0.86	4.31	0.90	2.50	1.18	1.30	0.93	1.30	0.90	2.53
2	0.61	3.40	0.65	1.68	0.69	0.71	0.68	0.71	0.65	1.71
4	0.46	2.67	0.50	1.18	0.54	0.50	0.54	0.50	0.50	1.19
8	0.46	2.60	0.49	1.14	0.53	0.49	0.53	0.49	0.49	1.14
mean	1.04	3.67	1.06	2.45	1.47	1.66	1.15	1.56	1.05	2.47

Table 4. Means of error ratios for extrapolated approximations, $n = 3$.

p	e_{De}	$e_{\mathrm{De'}}$	e_{MEe}	$e_{\mathrm{MEe'}}$	e_{R2e}	$e_{\mathrm{R2e'}}$	e_{Me}	$e_{\mathrm{Me'}}$	e_{MEs}	$e_{\mathrm{MEs'}}$
0	1.20	1.97	1.19	1.80	1.44	1.65	1.19	1.60	1.19	1.79
0.25	1.42	2.98	1.40	2.62	2.02	2.36	1.39	2.20	1.40	2.60
0.5	1.52	3.82	1.51	3.18	2.34	3.25	1.51	2.57	1.52	3.14
0.75	1.48	4.51	1.49	3.53	2.65	3.16	1.49	2.83	1.50	3.48
1	1.42	4.91	1.45	3.64	3.64	4.50	1.45	2.85	1.46	3.59
1.5	0.85	3.58	0.89	2.09	1.23	1.40	0.90	1.40	0.90	2.05
2	0.57	2.70	0.62	1.22	0.66	0.73	0.64	0.73	0.63	1.19
4	0.36	1.89	0.44	0.57	0.45	0.42	0.46	0.42	0.44	0.57
8	0.34	1.77	0.41	0.51	0.44	0.40	0.44	0.40	0.42	0.51
mean	1.02	3.13	1.04	2.13	1.65	1.98	1.05	1.66	1.05	2.10

We also computed $v_{\mathrm{maxD}} = v_{n,\alpha_0}$, choosing $n = n_{\mathrm{maxD}}$ as n_{D} in Theorem 1, and $v_{\mathrm{maxDe}} = v_{n,\alpha_0}$ with n as the nearest integer to $1.1 n_{\mathrm{maxD}}$.

In model equations the exact solutions are known. For each test we found α_{opt} as $\alpha_i \in \Omega$ with the smallest error: $\|u_{\alpha_{\mathrm{opt}}} - u_*\| = \min\{\|u_{\alpha_i} - u_*\|, \alpha_i \in \Omega\}$. All problems were solved 10 times, assuming that the noise level is $\delta = d\,\|f - f_\delta\|$ with $\delta \in \{0.3; 10^{-i},\ i = 1,\ \ldots,\ 6\}$ and with $d = 1,\ 2$. In tables the column corresponding to $d = 2$ is shown next to the column for $d = 1$ and we then use prime in the name of the rule. Tables 1–5 show the averages (over all problems, all δ and 10 runs) of error of the approximate solution by used parameter choice, divided by the smallest error of the Tikhonov approximation: for example $e_{\mathrm{D}} = \|u_{\alpha_{\mathrm{D}}} - u_*\| \,/\, \|u_{\alpha_{\mathrm{opt}}} - u_*\|$. Tables 1, 2 contain results for Tikhonov method, Tables 3, 4 for the extrapolated Tikhonov method with $n = 2,\ 3$, Table 5 for approximations v_{maxD}, v_{maxDe}. As can be seen from Table 2, the balancing principle with original large constants $c = 2$ in rules B1, B2 gives significantly larger error than rule B3 and rules B1*, B2* with smaller constants. However, the results for balancing principle were worse than results for rules MEe, R2e, Me in Table 1. As Tables 3–5 show, in case $u_* \in \mathcal{R}(A)$ the error of the extrapolated approximation was in most cases smaller than the error of the best single Tikhonov approximation. Table 5 shows the advantage of the approximation v_{maxDe} for large p.

Table 5. Means of error ratios for extrapolated approximations, $n = \max$.

p	e_{maxD}	$e_{\mathrm{maxD'}}$	e_{maxDe}	$e_{\mathrm{maxDe'}}$
0	1.34	2.39	1.22	2.12
0.25	1.63	3.76	1.44	3.25
0.5	1.74	5.03	1.54	4.28
0.75	1.62	6.17	1.49	5.15
1	1.43	6.74	1.42	5.57
1.5	0.96	4.92	0.86	4.30
2	0.63	3.84	0.56	3.38
4	0.32	2.66	0.31	2.49
8	0.27	2.34	0.27	2.24
mean	1.10	4.21	1.01	3.64

In Table 6, 7 averages of error ratios over δ and 10 runs of $v_{3,\alpha_{\mathrm{Me}}}$, v_{maxDe} for every problem are given. In most problems the ratios decreased by increasing p, especially for $p \geq 1$. Table 8 shows averages of error ratios over p and 10 runs in problem heat for every δ. More information about extrapolated Tikhonov method with numerical results can be found in [27].

Table 6. Means of error ratios for $v_{3,\alpha_{\mathrm{MEe}}}$ by problems.

p	baart	deriv2	foxgood	gravity	heat	ilaplace	phillips	shaw	spikes	wing	gauss	hilbert	lotkin	moler	pascal	prolate
0	1.39	1.00	1.33	1.05	1.01	1.02	1.00	1.55	1.12	1.47	1.10	1.24	1.22	1.07	1.02	1.48
0.25	2.05	1.00	1.38	1.03	0.99	1.13	0.98	1.46	1.25	2.60	1.07	1.34	1.26	1.09	2.08	1.65
0.5	2.34	1.00	1.32	0.97	0.96	1.11	0.93	1.28	1.22	2.97	1.01	1.29	1.21	1.06	4.25	1.40
0.75	1.83	0.97	1.28	0.91	0.94	1.06	0.87	1.19	1.12	2.48	0.93	1.21	1.11	1.01	6.09	1.18
1	1.58	0.93	1.12	0.83	0.90	0.99	0.79	0.99	1.00	2.23	0.83	1.11	1.01	0.94	7.44	0.98
1.5	0.97	0.78	0.77	0.66	0.78	0.81	0.60	0.79	0.85	1.17	0.64	0.90	0.77	0.77	2.46	0.70
2	0.70	0.62	0.55	0.55	0.65	0.66	0.48	0.62	0.70	0.66	0.53	0.72	0.62	0.57	0.78	0.57
4	0.42	0.39	0.39	0.42	0.44	0.44	0.39	0.40	0.48	0.48	0.44	0.45	0.41	0.32	0.66	0.51
8	0.38	0.35	0.38	0.40	0.40	0.41	0.39	0.39	0.41	0.48	0.43	0.41	0.36	0.31	0.66	0.51
mean	1.30	0.78	0.95	0.76	0.78	0.85	0.72	0.96	0.91	1.61	0.78	0.96	0.88	0.79	2.83	1.00

Tables 9, 10, 11 show error ratios for rules D, R2e in Tikhonov approximation u_α and rule R2e in extrapolated approximation $v_{2,\alpha}$ for values of $d = 0.5$, 0.6, 0.8, 1, 1.3, 1.6, 2, 3, 5. In case of overestimated noise level ($d > 1$) the rule R2e is significantly better than the discrepancy principle. In contrast to other rules, the rules R2, R2e also allow moderate underestimation of the noise level.

Table 7. Means of error ratios for v_{maxDe} by problems.

p	baart	deriv2	foxgood	gravity	heat	ilaplace	phillips	shaw	spikes	wing	gauss	hilbert	lotkin	moler	pascal	prolate
0	1.41	1.00	1.31	1.12	1.01	1.03	1.03	1.63	1.15	1.44	1.19	1.29	1.24	1.01	1.02	1.59
0.25	2.10	0.97	1.33	1.13	1.00	1.20	0.99	1.55	1.36	2.61	1.18	1.46	1.33	0.99	1.99	1.91
0.5	2.39	0.94	1.27	1.04	0.96	1.22	0.93	1.34	1.34	3.04	1.12	1.43	1.27	0.94	3.93	1.66
0.75	1.84	0.87	1.23	0.97	0.89	1.20	0.86	1.24	1.26	2.54	1.03	1.38	1.13	0.87	5.47	1.46
1	1.59	0.80	1.05	0.87	0.83	1.12	0.76	1.02	1.12	2.16	0.92	1.25	1.01	0.79	6.61	1.26
1.5	0.93	0.63	0.67	0.65	0.67	0.88	0.56	0.76	0.90	1.06	0.66	1.00	0.73	0.62	2.25	0.91
2	0.62	0.46	0.44	0.49	0.52	0.66	0.43	0.58	0.69	0.53	0.51	0.75	0.53	0.42	0.63	0.70
4	0.26	0.26	0.24	0.30	0.28	0.34	0.31	0.28	0.37	0.26	0.34	0.33	0.26	0.18	0.52	0.54
8	0.22	0.21	0.20	0.25	0.23	0.27	0.27	0.25	0.27	0.22	0.30	0.26	0.20	0.16	0.52	0.52
mean	1.26	0.68	0.86	0.76	0.71	0.88	0.68	0.96	0.94	1.54	0.81	1.02	0.86	0.66	2.55	1.17

Table 8. Means (over all p) of error ratios and errors for problem heat.

δ	e_{D}	$e_{\mathrm{D'}}$	e_{Me}	$e_{\mathrm{Me'}}$	e_{2MEe}	$e_{\mathrm{2MEe'}}$	e_{3De}	$e_{\mathrm{3De'}}$	$\lVert u_{\alpha_{\mathrm{opt}}} - u_* \rVert$
0.3	1.06	3.20	1.09	1.11	0.95	1.36	0.82	1.48	2.68e-001
10^{-1}	1.11	2.26	1.10	1.06	0.90	1.43	0.80	2.30	1.80e-001
10^{-2}	1.30	1.35	1.11	1.06	0.83	1.11	0.75	1.56	9.07e-002
10^{-3}	1.55	1.18	1.13	1.07	0.79	0.97	0.72	1.19	5.67e-002
10^{-4}	1.84	1.30	1.13	1.06	0.77	0.87	0.70	1.00	4.35e-002
10^{-5}	2.44	1.58	1.15	1.06	0.76	0.80	0.69	0.87	3.77e-002
10^{-6}	3.33	2.21	1.19	1.07	0.76	0.76	0.70	0.79	3.36e-002
mean	1.81	1.87	1.13	1.07	0.82	1.05	0.74	1.31	

Table 9. Means of error ratios for $u_{\alpha_{\mathrm{D}}}$.

$p \setminus d$	1	1.3	1.6	2	3	5
0	1.26	1.76	1.98	2.21	2.55	3.03
0.25	1.46	2.42	2.85	3.28	3.92	4.83
0.5	1.56	2.89	3.53	4.20	5.20	6.70
0.75	1.59	3.23	4.04	4.93	6.19	8.21
1	1.82	3.15	4.02	4.98	6.34	8.57
1.5	2.47	2.58	3.17	3.95	4.92	6.78
2	2.73	2.63	3.14	3.86	4.70	6.40
4	2.92	2.68	3.15	3.85	4.65	6.29
8	2.95	2.68	3.15	3.85	4.64	6.26
mean	2.08	2.67	3.23	3.90	4.79	6.34

Table 10. Means of error ratios for $u_{\alpha_{\mathrm{R2e}}}$.

$p \setminus d$	0.5	0.6	0.8	1	1.3	1.6	2	3	5
0	1.80	1.44	1.38	1.42	1.48	1.54	1.61	1.82	2.14
0.25	2.25	1.83	1.78	1.90	2.04	2.18	2.31	2.65	3.14
0.5	2.60	2.05	1.98	2.09	2.25	2.39	2.57	3.16	4.25
0.75	2.68	2.04	2.05	2.17	2.43	2.77	3.01	3.59	4.44
1	2.71	2.21	2.12	2.31	2.46	2.63	2.88	3.39	5.20
1.5	1.85	1.37	1.37	1.40	1.46	1.54	1.67	2.01	2.71
2	1.61	1.20	1.15	1.15	1.17	1.22	1.31	1.56	2.13
4	1.48	1.19	1.15	1.13	1.14	1.18	1.25	1.45	1.96
8	1.46	1.19	1.14	1.13	1.14	1.18	1.24	1.45	1.94
mean	2.05	1.61	1.57	1.63	1.73	1.85	1.98	2.34	3.10

Table 11. Means of error ratios for $v_{2,\alpha_{\mathrm{R2e}}}$.

$p \setminus d$	0.5	0.6	0.8	1	1.3	1.6	2	3	5
0	112	15.3	1.96	1.48	1.42	1.45	1.52	1.66	1.94
0.25	182	21.7	2.49	1.99	1.97	2.06	2.16	2.39	2.79
0.5	228	24.4	2.76	2.15	2.11	2.19	2.34	3.28	3.82
0.75	257	25.3	3.21	2.47	2.46	2.68	2.85	3.20	3.73
1	262	24.4	2.93	2.26	2.27	2.36	3.11	4.06	5.15
1.5	272	19.3	1.44	1.18	1.21	1.25	1.30	1.46	1.84
2	239	12.4	0.77	0.69	0.68	0.69	0.71	0.79	1.03
4	236	4.36	0.56	0.54	0.52	0.51	0.50	0.51	0.62
8	236	3.82	0.55	0.53	0.51	0.50	0.49	0.50	0.60
mean	225	16.8	1.85	1.47	1.46	1.52	1.66	1.98	2.39

References

[1] F. Bauer and T. Hohage. A Lepskij-type stopping rule for regularized Newton methods. *Inverse Problems*, **21**(6):1975–1991, 2005. Doi:10.1088/0266-5611/21/6/011.

[2] F. Bauer, P. Mathé and S. Pereverzev. Local solutions to inverse problems in geodesy: The impact of the noise covariance structure upon the accuracy of estimation. *J. of Geodesy*, **81**(1):39–51, 2007. Doi:10.1007/s00190-006-0049-5.

[3] F. Bauer and A. Munk. Optimal regularization for ill-posed problems in metric spaces. *J. Inverse and Ill-Posed Problems*, **15**(2):137–148, 2007. Doi:10.1515/JIIP.2007.007.

[4] F. Bauer and S. Pereverzev. Regularization without preliminary knowledge of smoothness and error behaviour. *European J. Appl. Math.*, **16**(3):303–317, 2005. Doi:10.1017/S0956792505006236.

[5] C. Brezinski, M. Redivo-Zaglia, G. Rodriguez and S. Seatzu. Extrapolation techniques for ill-conditioned linear systems. *Numer. Math.*, **81**(1):1–29, 1998. Doi:10.1007/s002110050382.

[6] C. Brezinski, M. Redivo-Zaglia, G. Rodriguez and S. Seatzu. Multi-parameter

regularization techniques for ill-conditioned linear systems. *Numer. Math.*, **94**(2):203–228, 2003. Doi:10.1007/s00211-002-0435-8.

[7] C. Brezinski, G. Rodriguez and S. Seatzu. Error estimates for linear systems with applications to regularization. *Numer. Algor.*, **49**:85–104, 2008. Doi:10.1007/s11075-008-9163-1.

[8] H. Cao, M. Klibanov and S. V. Pereverzev. A Carleman estimate and the balancing principle in the quasi-reversibility method for solving the Cauchy problem for the Laplace equation. *Inverse Problems*, **25**(3):035005 (21 pp), 2009.

[9] H. Gfrerer. An a posteriori parameter choice for ordinary and iterated Tikhonov regularization of ill-posed problems leading to optimal convergence rates. *Math. Comp.*, **49**(180):507–522, 1987. Doi:10.2307/2008325.

[10] C. W. Groetsch and J. T. King. Extrapolation and the method of regularization for generalized inverses. *J. Approx. Theory*, **25**(3):233–247, 1979. Doi:10.1016/0021-9045(79)90014-5.

[11] U. Hämarik, R. Palm and T. Raus. Use of extrapolation in regularization methods. *J. Inverse and Ill-Posed Problems*, **15**(3):277–294, 2007. Doi:10.1515/jiip.2007.015.

[12] U. Hämarik, R. Palm and T. Raus. Extrapolation of Tikhonov and Lavrentiev regularization methods. *Journal of Physics: Conference Series*, **135**:(012048):(8 pp), 2008.

[13] U. Hämarik and T. Raus. About the balancing principle for choice of the regularization parameter. *Numerical Functional Analysis and Optimization*, **30**(9&10):951–970, 2009.

[14] M. Hanke and C. W. Groetsch. Nonstationary iterated Tikhonov regularization. *J. Optim. Theory Appl.*, **98**(1):37–53, 1998. Doi:10.1023/A:1022680629327.

[15] P. C. Hansen. Regularization tools: A Matlab package for analysis and solution of discrete ill-posed problems. *Numerical Algorithms*, **6**(1):1–35, 1994. Doi:10.1007/BF02149761.

[16] D. C. Joyce. Survey of extrapolation processes in numerical analysis. *SIAM Rev.*, **13**:435–490, 1971. Doi:10.1137/1013092.

[17] R. D. Lazarov, S. Lu and S. V. Pereverzev. On the balancing principle for some problems of numerical analysis. *Numerische Mathematik*, **106**:659–689, 2007. Doi:10.1007/s00211-007-0076-z.

[18] S. Lu and S. V. Pereverzev. Numerical differentiation from a viewpoint of regularization theory. *Math. Comp.*, **75**(256):1853–1870, 2006. Doi:10.1090/S0025-5718-06-01857-6.

[19] S. Lu and S. V. Pereverzev. Sparse recovery by the standard Tikhonov method. *Numer. Math*, **112**:403–424, 2009. Doi:10.1007/s00211-009-0214-x.

[20] S. Lu, S. V. Pereverzev and R. Ramlau. An analysis of Tikhonov regularization for nonlinear ill-posed problems under a general smoothness assumption. *Inverse Problems*, **23**(1):217–230, 2007. Doi:10.1088/0266-5611/23/1/011.

[21] G. I. Marchuk and V. V. Shaidurov. *Difference methods and their extrapolations.* Springer-Verlag, New York, 1983.

[22] P. Mathé. The Lepskii principle revisited. *Inverse Problems*, pp. L11–L15, 2006. Doi:10.1088/0266-5611/22/3/L02.

[23] P. Mathé and S. V. Pereverzev. Geometry of linear ill-posed problems in variable Hilbert scales. *Inverse Problems*, **19**(3):789–803, 2003. Doi:10.1088/0266-5611/19/3/319.

[24] P. Mathé and S. V. Pereverzev. The discretized discrepancy principle under general source conditions. *Journal of Complexity*, **22**(3):371–381, 2006. Doi:10.1016/j.jco.2005.06.008.

[25] P. Mathé and S. V. Pereverzev. Regularization of some linear ill-posed problems with discretized random noisy data. *Math. Comp.*, **75**(256):1913–1929, 2006. Doi:10.1090/S0025-5718-06-01873-4.

[26] V. A. Morozov. On the solution of functional equations by the method of regularization. *Soviet Math. Dokl.*, **7**:414–417, 1966.

[27] R. Palm. *Numerical comparison of regularization algorithms for solving ill-possed problems.* University of Tartu, 2010. PhD thesis

[28] S. V. Pereverzev and E. Schock. On the adaptive selection of the parameter in the regularization of ill-posed problems. *SIAM J. Numerical Analysis*, **43**(5):2060–2076, 2005. Doi:10.1137/S0036142903433819.

[29] T. Raus. The principle of the residual in the solution of ill-posed problems with non-selfadjoint operator. *Acta et comment. Univ. Tartuensis*, **715**:12–20, 1985. (In Russian)

[30] T. Raus and U. Hämarik. On the quasioptimal regularization parameter choices for solving ill-posed problems. *J. Inverse Ill-Posed Problems*, **15**(4):419–439, 2007. Doi:10.1515/jiip.2007.023.

[31] T. Raus and U. Hämarik. New rule for choice of the regularization parameter in (iterated) Tikhonov method. *Math. Model. Anal.*, **14**(2):187–198, 2009. Doi:10.3846/1392-6292.2009.14.187-198.

[32] T. Raus and U. Hämarik. On numerical realization of quasioptimal parameter choices in (iterated) Tikhonov and Lavrentiev regularization. *Math. Model. Anal.*, **14**(1):99–108, 2009. Doi:10.3846/1392-6292.2009.14.99-108.

[33] G. Rodriguez, S. Seatzu and D. Theis. A new technique for ill-conditioned linear systems. *Numer. Algorithms*, **33**((1–4)):433–442, 2003.

[34] V. V. Saidurov. Continuation with respect to the parameter in the method of regularization. In *Numerical methods of linear algebra*, pp. 77–85. Nauka, Novosibirsk, 1973. In Russian

[35] U. Tautenhahn and U. Hämarik. The use of monotonicity for choosing the regularization parameter in ill-posed problems. *Inverse Problems*, **15**(6):1487–1505, 1999. Doi:10.1088/0266-5611/15/6/307.

[36] G. M. Vainikko and A. Yu. Veretennikov. *Iteration Procedures in Ill-Posed Problems.* Nauka, Moscow, 1986. (In Russian)