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A Plankton Allelopathic Model Described by a Delayed Quasilinear Parabolic System*

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Abstract. The quasilinear parabolic system has been applied to a variety of physical and engineering problems. However, most works lack effective techniques to deal with the asymptotic stability. This paper is concerned with the existence and stability of solutions for a plankton allelopathic model described by a quasilinear parabolic system, in which the diffusions are density-dependent. By the coupled upper and lower solutions and its associated monotone iterations, it is shown that under some parameter conditions the positive uniform equilibrium is asymptotically stable. Some biological interpretations for our results are given.

Keywords: parabolic reaction–diffusion equation, Volterra equation, quasi-linearization, global stability, existence.

AMS Subject Classification: 35B35; 92D50.

1 Introduction

We consider a coupled system of quasilinear parabolic equations in a bounded domain. The system of equations is given in the form

$$\begin{cases} \partial u_1/\partial t - \nabla\cdot\big(d_1(u_1/n_1)^m\nabla u_1\big) = u_1(a_1 - b_{11}u_1 - b_{12}u_2 - e_1u_1u_2) \\ \quad (t>0,\ x\in\Omega), \\ \partial u_2/\partial t - \nabla\cdot\big(d_2(u_2/n_2)^m\nabla u_2\big) = u_2\big(a_2 - b_{21}u_1 - b_{22}u_2 - e_2(u_1)_\tau u_2\big) \\ \quad (t>0,\ x\in\Omega), \\ \partial u_1/\partial\nu = \partial u_2/\partial\nu = 0 \quad (t>0,\ x\in\partial\Omega), \\ u_i(t,x) = \psi_i(t,x) \quad \big(t\in[-\tau,0],\ x\in\Omega\big),\ i=1,2, \end{cases} \tag{1.1}$$

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where Ω is a bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega$, $\partial/\partial\nu$ denotes the outward normal derivative on $\partial\Omega$. It is assumed that the boundary $\partial\Omega$ is of class $C^{1+\alpha}$. $\psi_i(t,x) \in C^{\alpha/2,\alpha}([-\tau,0] \times \overline{\Omega})$ has a positive lower bound and satisfies the compatibility condition. $(u_1)_\tau \equiv u_1(t-\tau,x)$ is the discrete delay term. The density-dependent diffusion coefficient $D_i(u_i) \equiv d_i(u_i/n_i)^m$ $(i = 1,2)$, in which $n_i > 0$, $d_i > 0$ and $m \geq 0$, has the property $D_i(0) = 0$, which means the elliptic operators are degenerate. In the case $m = 0$, (1.1) is reduced to the semilinear parabolic system.

The system (1.1) is described by the plankton allelopathic competition model in aquatic ecology. u_1, u_2 stand for the population density (number of cells per liter) of two competing species; a_1, a_2 are the rates of cell proliferation per hour; b_{11}, b_{22} are the rates of intra-specific competition of the first and the second species, respectively; b_{12}, b_{21} are the rates of inter-specific competition of the first and the second species, respectively; a_i/b_{ii} $(i = 1,2)$ are environmental carrying capacities (representing number of cell per liter). e_1 and e_2 are, respectively, the rates of toxic inhibition of the first species by the second and vice verse. The units of a_i, b_{ij} and e_i are per hour per cell and the unit of time is hours. The planktonic allelopathy model is investigated by ordinary differential equations [2, 9, 12, 11] and semilinear parabolic system [3, 23, 24, 25, 26]. Moreover, we incorporate the effect of disperse described by $D_i(u_i) = d_i(u_i/n_i)^m$ into the previous planktonic allelopathy model. Aikman and Hewitt have observed that the coefficients of diffusion are increasing when the densities of the populations are increasing in the experiment of dispersal patterns for grasshoppers [1]. Murray has first used the form $D(u) = d(u/n)^m$ to describe the density-dependent diffusion. The schematic solution obtained in [13] coincides with the experimental observation. Pao and Ruan have first introduced the density-dependent diffusion $D(u) = (n-1)u^{m-1}$ into Lotka–Volterra model [20]. They have investigated the existence, uniqueness and asymptotic behaviour of positive time-dependent solutions for the quasilinear parabolic system with quasimonotone nondecreasing reaction functions.

However, the requirement of the reaction functions in [19, 20, 21] are quasimonotone nondecreasing. The condition is relaxed in this paper to mixed quasimonotone nondecreasing reaction functions, which leads to the difficult point that the ordered upper and lower solutions do not exist. To overcome it, we construct the coupled upper and lower solutions. In this paper, we aim to study the existence and asymptotic behaviour of (1.1) by the method of coupled upper and lower solutions.

The rest of this paper is organized as follows. In Section 2 we extend the monotone iteration method of [20] for quasimonotone nondecreasing reaction functions to delayed quasimonotone nondecreasing reaction functions. We show the existence and asymptotic stability of (1.1). In Section 3, we give the detail parameter conditions such that the positive uniform equilibrium is asymptotically stable. Section 4 is devoted to some discussions about the biological and mathematical senses.

2 Monotone Iteration Method

In this section, we extend the monotone iteration method of [20] to delayed reaction functions. For the simplicity, throughout this paper, we denote

$$D = [0, \infty) \times \Omega, \qquad \overline{D} = [0, \infty) \times \overline{\Omega}, \qquad S = (0, \infty) \times \partial\Omega,$$
$$D_0^{(i)} = [-\tau_i, 0] \times \Omega, \qquad Q^{(i)} = [-\tau, \infty) \times \Omega,$$
$$D_0 = D_0^{(1)} \times D_0^{(2)}, \qquad Q = Q^{(1)} \times Q^{(2)},$$

and let $C^m(Q)$, $C^\alpha(Q)$ be the respective space of $m-$times differentiable and Hölder continuous functions in Q, where Q represents a domain. For vector functions with $N-$components we denote the above function space by $\mathcal{C}^m(Q)$ and $\mathcal{C}^\alpha(Q)$, respectively. We also denote

$$f_1(u_1, u_2) \equiv u_1(a_1 - b_{11}u_1 - b_{12}u_2 - e_1u_1u_2),$$
$$f_2\big(u_1, u_2, (u_1)_\tau\big) \equiv u_2(a_2 - b_{21}u_1 - b_{22}u_2 - e_2(u_1)_\tau u_2).$$

Definition 1. A pair of function $\tilde{\mathbf{u}} = (\tilde{u}_1, \tilde{u}_2), \hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2) \in \mathcal{C}(\overline{Q}) \cap \mathcal{C}^{1,2}(Q)$ are called coupled upper and lower solutions of (1.1) if $\hat{\mathbf{u}}$ has a positive lower bound, and $\tilde{\mathbf{u}} \geq \hat{\mathbf{u}}$,

$$\begin{aligned}
&\partial\tilde{u}_1/\partial t - \nabla \cdot \big(D_1(\tilde{u}_1)\nabla\tilde{u}_1\big) \geq f_1(\tilde{u}_1, \hat{u}_2) \quad \text{in } D,\\
&\partial\tilde{u}_2/\partial t - \nabla \cdot \big(D_2(\tilde{u}_2)\nabla\tilde{u}_2\big) \geq f_2\big(\hat{u}_1, \tilde{u}_2, (\hat{u}_1)_\tau\big) \quad \text{in } D,\\
&\partial\hat{u}_1/\partial t - \nabla \cdot \big(D_1(\hat{u}_1)\nabla\hat{u}_1\big) \leq f_1(\hat{u}_1, \tilde{u}_2) \quad \text{in } D,\\
&\partial\hat{u}_2/\partial t - \nabla \cdot \big(D_2(\hat{u}_2)\nabla\hat{u}_2\big) \leq f_2\big(\tilde{u}_1, \hat{u}_2, (\tilde{u}_1)_\tau\big) \quad \text{in } D,\\
&\partial\tilde{u}_i/\partial\nu \geq 0, \qquad \partial\hat{u}_i/\partial\nu \leq 0, \quad i = 1, 2, \text{ on } S,\\
&\tilde{u}_i(t, x) \geq \psi_i(t, x), \qquad \hat{u}_i(t, x) \leq \psi_i(t, x), \quad i = 1, 2, \text{ in } D_0^{(i)}.
\end{aligned} \tag{2.1}$$

Define a modified function $\overline{D}_i(u_i)$ by

$$\overline{D}_i(u_i) = \begin{cases} D_i(u_i) + (u_i - \tilde{u}_i) & \text{if } u_i > \tilde{u}_i,\\ D_i(u_i) & \text{if } \hat{u}_i \leq u_i \leq \tilde{u}_i,\\ D_i(u_i) + (\hat{u}_i - u_i) & \text{if } u_i < \hat{u}_i. \end{cases} \tag{2.2}$$

Then there exists $d_0 > 0$ such that $\overline{D}_i(u) \geq d_0$ for all $u \in \mathbb{R}$. Define

$$w_i = I_i(u_i) = \int_0^{u_i} \overline{D}_i(s)\, ds \quad \text{for } u_i \geq 0,\ i = 1, 2, \tag{2.3}$$

Derivativing (2.3), we have

$$I_i'(u_i) = dI_i/du_i = \overline{D}_i(u_i) > 0.$$

Then the inverse $u_i \equiv q_i(w_i)$ exists and is an increasing function of $w_i > 0$.

For a given pair of coupled upper and lower solutions $\tilde{\mathbf{u}}$, $\hat{\mathbf{u}}$ we set

$$\Lambda_i = \{u_i \in C(\overline{Q})\colon \hat{u}_i \le u_i \le \tilde{u}_i\}, \qquad \Lambda = \{\mathbf{u} \in \mathcal{C}(\overline{Q})\colon \hat{\mathbf{u}} \le \mathbf{u} \le \tilde{\mathbf{u}}\},$$
$$\Lambda \times \bar{\Lambda} = \{(\mathbf{u}, \mathbf{w}) \in \mathcal{C}(\overline{Q}) \times \mathcal{C}(\overline{Q})\colon (\hat{\mathbf{u}}, \hat{\mathbf{w}}) \le (\mathbf{u}, \mathbf{w}) \le (\tilde{\mathbf{u}}, \tilde{\mathbf{w}})\},$$

where $\tilde{w}_i = I_i(\tilde{u}_i)$ and $\hat{w}_i = I(\hat{u}_i)$. There exist smooth nonnegative functions $\beta_i \equiv \beta_i(t,x)$ such that

$$\beta_i \overline{D}_i(u_i) + \frac{\partial f_i}{\partial u_i}(\cdot, \mathbf{u}) \ge 0, \quad \text{for } \mathbf{u} \in \Lambda. \tag{2.4}$$

In fact, it suffices to choose any $\beta_i(t,x)$ satisfying

$$\beta_i(t,x) \ge \max\{-(\partial f_i/\partial u_i(t,x,\mathbf{u}))/\overline{D}_i(u_i)\colon \mathbf{u} \in \Lambda\}.$$

Define for each $i = 1, 2$,

$$\begin{aligned} F_1(t,x,u_1,u_2) &= \beta_1(t,x) I_1(u_1) + f_1(t,x,u_1,u_2), \\ F_2(t,x,u_1,u_2) &= \beta_2(t,x) I_2(u_2) + f_2(t,x,u_1,u_2,(u_1)_\tau), \\ L_i w_i &= \nabla \cdot (\nabla w_i) - \beta_i(t,x) w_i. \end{aligned} \tag{2.5}$$

Since (2.4), (2.5) and $I_i'(u_i) = \overline{D}_i(u_i)$, $F_i(\cdot, u_i, u_j)$ possess the monotone property

$$F_i(\cdot, v_i, u_j) \le F_i(\cdot, u_i, v_j) \quad \text{whenever} \quad \hat{\mathbf{u}} \le \mathbf{v} \le \mathbf{u} \le \tilde{\mathbf{u}}. \tag{2.6}$$

We consider the system

$$\begin{aligned} &(\overline{D}_i(u_i))^{-1} \partial w_i/\partial t - L_i w_i = F_i(t,x,u_i,u_j) \quad \text{in } D, \\ &\partial w_i/\partial \nu = 0 \quad \text{on } S, \quad w_i(t,x) = \eta_i(t,x) \quad \text{in } D_0^{(i)}, \\ &u_i = q_i(w_i) \quad \text{for } i = 1,2 \text{ in } \overline{Q}^{(i)}, \end{aligned} \tag{2.7}$$

where $\eta_i(t,x) = I_i(\psi_i(t,x))$. Note that if $\overline{D}_i(u_i) = D_i(u_i)$, then (2.7) is equivalent to (1.1).

By using $\underline{\mathbf{u}}^{(0)} = \hat{\mathbf{u}}$ and $\overline{\mathbf{u}}^{(0)} = \tilde{\mathbf{u}}$ as the initial iterations we can construct sequences $\{\underline{\mathbf{u}}^{(m)}, \underline{\mathbf{w}}^{(m)}\}$ and $\{\overline{\mathbf{u}}^{(m)}, \overline{\mathbf{w}}^{(m)}\}$ from the nonlinear iteration process

$$\begin{aligned} &(\overline{D}_i(\overline{u}_i^{(m)}))^{-1} \partial \overline{w}_i^{(m)}/\partial t - L_i \overline{w}_i^{(m)} = F_i(t,x,\overline{u}_i^{(m-1)}, \underline{u}_j^{(m-1)}), \\ &(\overline{D}_i(\underline{u}_i^{(m)}))^{-1} \partial \underline{w}_i^{(m)}/\partial t - L_i \underline{w}_i^{(m)} = F_i(t,x,\underline{u}_i^{(m-1)}, \overline{u}_j^{(m-1)}) \\ &\quad \text{for } i = 1,\ j = 2;\ \text{or } i = 2,\ j = 1;\ \text{in } D, \\ &\partial \overline{w}_i^{(m)}/\partial \nu = \partial \underline{w}_i^{(m)}/\partial \nu = 0 \quad \text{for } i = 1,2 \text{ on } S, \\ &\overline{u}_i^{(m)} = q_i(\overline{w}_i^{(m)}), \qquad \underline{u}_i^{(m)} = q_i(\underline{w}_i^{(m)}) \quad \text{for } i = 1,2 \text{ in } \overline{Q}^{(i)}, \\ &\overline{w}_i^{(m)}(t,x) = \underline{w}_i^{(m)}(t,x) = \eta_i(t,x) \quad \text{for } i = 1,2 \text{ in } D_0^{(i)}. \end{aligned} \tag{2.8}$$

The sequences $\{\underline{\mathbf{u}}^{(m)}, \underline{\mathbf{w}}^{(m)}\}$ and $\{\overline{\mathbf{u}}^{(m)}, \overline{\mathbf{w}}^{(m)}\}$ are well defined by the existence theorem of [8]. The following lemma gives the monotone property of these sequences.

Lemma 1. *The sequences* $\{\overline{\mathbf{u}}^{(m)}, \overline{\mathbf{w}}^{(m)}\}$, $\{\underline{\mathbf{u}}^{(m)}, \underline{\mathbf{w}}^{(m)}\}$ *governed by* (2.8) *possess the monotone property*

$$(\hat{\mathbf{u}}, \hat{\mathbf{w}}) \le \big(\underline{\mathbf{u}}^{(\mathbf{m})}, \underline{\mathbf{w}}^{(\mathbf{m})}\big) \le \big(\underline{\mathbf{u}}^{(\mathbf{m+1})}, \underline{\mathbf{w}}^{(\mathbf{m+1})}\big) \le \big(\overline{\mathbf{u}}^{(\mathbf{m+1})}, \overline{\mathbf{w}}^{(\mathbf{m+1})}\big) \\ \le \big(\overline{\mathbf{u}}^{(\mathbf{m})}, \overline{\mathbf{w}}^{(\mathbf{m})}\big) \le (\tilde{\mathbf{u}}, \tilde{\mathbf{w}}) \quad \text{for } m = 1, 2, \ldots. \tag{2.9}$$

Moreover, for each $m = 1, 2, \ldots$, $\overline{\mathbf{u}}^{(\mathbf{m})}$ *and* $\underline{\mathbf{u}}^{(\mathbf{m})}$ *are coupled upper and lower solutions of* (1.1).

Proof. Let $\underline{z}_i^{(1)} = \underline{w}_i^{(1)} - \underline{w}_i^{(0)}$, $i = 1, 2$. Then by (2.1) and (2.8), $\underline{z}_i^{(1)}$ satisfies

$$\begin{aligned}
&\big(\overline{D}_i\big(\underline{u}_i^{(1)}\big)\big)^{-1}\partial \underline{z}_i^{(1)}/\partial t - L_i \underline{z}_i^{(1)} \\
&\quad = F_i\big(\cdot, \underline{u}_i^{(0)}, \overline{u}_j^{(0)}\big) - \big[\big(\overline{D}_i\big(\underline{u}_i^{(1)}\big)\big)^{-1}\partial \underline{w}_i^{(0)}/\partial t - L_i \underline{w}_i^{(0)}\big] \\
&\quad = F_i\big(\cdot, \underline{u}_i^{(0)}, \overline{u}_j^{(0)}\big) - \big[\big(\overline{D}_i\big(\underline{u}_i^{(0)}\big)\big)^{-1}\partial \underline{w}_i^{(0)}/\partial t - L_i \underline{w}_i^{(0)}\big] \\
&\qquad - \big[\big(\overline{D}_i\big(\underline{u}_i^{(1)}\big)\big)^{-1} - \big(\overline{D}_i\big(\underline{u}_i^{(0)}\big)\big)^{-1}\big]\partial \underline{w}_i^{(0)}/\partial t \\
&\quad \ge -\big[\big(\overline{D}_i\big(\underline{u}_i^{(1)}\big)\big)^{-1} - \big(\overline{D}_i\big(\underline{u}_i^{(0)}\big)\big)^{-1}\big]\partial \underline{w}_i^{(0)}/\partial t.
\end{aligned}$$

Since by the mean value theorem,

$$\begin{aligned}
\big(\overline{D}_i\big(\underline{u}_i^{(1)}\big)\big)^{-1} - \big(\overline{D}_i\big(\underline{u}_i^{(0)}\big)\big)^{-1} &= -\big[\,\overline{D}_i'\big(\xi^{(0)}\big)/\big(\overline{D}_i\big(\xi^{(0)}\big)\big)^2\big]\big(\underline{u}_i^{(1)} - \underline{u}_i^{(0)}\big) \\
&= -\big[\,\overline{D}_i'\big(\xi^{(0)}\big)/\big(\overline{D}_i\big(\xi^{(0)}\big)\big)^3\big]\big(\underline{w}_i^{(1)} - \underline{w}_i^{(0)}\big),
\end{aligned}$$

for some intermediate value $\xi^{(0)} \equiv \xi^{(0)}(t, x)$ between $\underline{u}_i^{(0)}$ and $\underline{u}_i^{(1)}$, we have

$$\begin{aligned}
&\big(\overline{D}_i\big(\underline{u}_i^{(1)}\big)\big)^{-1}\partial \underline{z}_i^{(1)}/\partial t - L_i \underline{z}_i^{(1)} + \gamma^{(0)}\underline{z}_i^{(1)} \ge 0, \\
&\gamma^{(0)} = -\big[\,\overline{D}_i'\big(\xi^{(0)}\big)/\big(\overline{D}_i\big(\xi^{(0)}\big)\big)^3\big]\partial \underline{w}_i^{(0)}/\partial t.
\end{aligned} \tag{2.10}$$

And the boundary and initial inequalities satisfy

$$\partial \underline{z}_i^{(1)}/\partial \nu = 0 \quad \text{on } S, \quad \underline{z}_i^{(1)}(0, x) = \eta_i(0, x) - \eta_i(0, x) = 0 \quad \text{in } \Omega. \tag{2.11}$$

In view of the definition of $\overline{D}_i$ in (2.2), the function $\overline{D}_i(\underline{u}_i^{(1)})\gamma^{(0)}$ of (2.10) is bounded. By Lemma 2.1 of [20] $\underline{z}_i^{(1)} \ge 0$ on $\overline{D}$. This gives $\underline{w}_i^{(1)} \ge \underline{w}_i^{(0)}$ and thus $\underline{u}_i^{(1)} \ge \underline{u}_i^{(0)}$. A similar argument yields $\overline{w}_i^{(1)} \le \overline{w}_i^{(0)}$ and $\overline{u}_i^{(1)} \le \overline{u}_i^{(0)}$.

Moreover, letting $z_i^{(1)} = \overline{w}_i^{(1)} - \underline{w}_i^{(1)}$, by (2.8), and after the similar above argument

$$\begin{aligned}
&\big(\overline{D}_i\big(\overline{u}_i^{(1)}\big)\big)^{-1}\partial z_i^{(1)}/\partial t - L_i z_i^{(1)} + \gamma_i^{(0)} z_i^{(1)} \\
&\quad = F_i\big(\cdot, \overline{u}_i^{(0)}, \underline{u}_j^{(0)}\big) - F_i\big(\cdot, \underline{u}_i^{(0)}, \overline{u}_j^{(0)}\big) \ge 0 \quad \text{in } D, \\
&\partial z_i^{(1)}/\partial \nu = 0 \quad \text{on } S, \quad z_i^{(1)}(0, x) = \eta_i(0, x) - \eta_i(0, x) = 0 \quad \text{in } \Omega,
\end{aligned}$$

where $\gamma_i^{(0)} = -\big[\,\overline{D}_i'\big(\xi_i^{(0)}\big)/\big(\overline{D}_i\big(\xi_i^{(0)}\big)\big)^3\big]\partial \underline{w}_i^{(0)}/\partial t$, for some intermediate value $\xi_i^{(0)} \equiv \xi_i^{(0)}(t, x)$ between $\underline{u}_i^{(0)}$ and $\overline{u}_i^{(0)}$. It follows again from Lemma 2.1 of [20] that $\overline{\mathbf{w}}^{(1)} \ge \underline{\mathbf{w}}^{(1)}$ and thus $\overline{\mathbf{u}}^{(1)} \ge \underline{\mathbf{u}}^{(1)}$. The above conclusions show that

$$\big(\underline{\mathbf{u}}^{(0)}, \underline{\mathbf{w}}^{(0)}\big) \le \big(\underline{\mathbf{u}}^{(1)}, \underline{\mathbf{w}}^{(1)}\big) \le \big(\overline{\mathbf{u}}^{(1)}, \overline{\mathbf{w}}^{(1)}\big) \le \big(\overline{\mathbf{u}}^{(0)}, \overline{\mathbf{w}}^{(0)}\big). \tag{2.12}$$

Now we show that $\overline{\mathbf{u}}^{(1)}$ and $\underline{\mathbf{u}}^{(1)}$ are coupled upper and lower solutions of (1.1). Since (2.2) and (2.12), $\overline{D}_i(\overline{u}_i^{(1)}) = D_i(\overline{u}_i^{(1)})$ for $i = 1, 2$. It suffices to show $\overline{\mathbf{u}}^{(1)}$ and $\underline{\mathbf{u}}^{(1)}$ satisfy (2.1). Since (2.6), (2.8) and (2.11), we have

$$\begin{aligned}
&\big(D_i\big(\overline{u}_i^{(1)}\big)\big)^{-1}\partial\overline{w}_i^{(1)}/\partial t - L_i\overline{w}_i^{(1)} = F_i\big(\cdot, \overline{u}_i^{(0)}, \underline{u}_j^{(0)}\big) \geq F_i\big(\cdot, \overline{u}_i^{(1)}, \underline{u}_j^{(1)}\big),\\
&\big(D_i\big(\underline{u}_i^{(1)}\big)\big)^{-1}\partial\underline{w}_i^{(1)}/\partial t - L_i\underline{w}_i^{(1)} = F_i\big(\cdot, \cdot, \underline{u}_i^{(0)}, \overline{u}_j^{(0)}\big) \leq F_i\big(\cdot, \underline{u}_i^{(1)}, \overline{u}_j^{(1)}\big),\\
&\partial\overline{u}_i^{(1)}/\partial\nu = \partial\underline{u}_i^{(1)}/\partial\nu = 0, \quad \overline{u}_i^{(1)}(0,x) = \eta_i(0,x), \quad \underline{u}_i^{(1)}(0,x) = \eta_i(0,x).
\end{aligned}$$

Next we use an induction method. By choosing $\overline{\mathbf{u}}^{(1)}$ and $\underline{\mathbf{u}}^{(1)}$ as the coupled upper and lower solutions $\tilde{\mathbf{u}}$ and $\hat{\mathbf{u}}$, after the similar above argument, we have

$$\big(\underline{\mathbf{u}}^{(1)}, \underline{\mathbf{w}}^{(1)}\big) \leq \big(\underline{\mathbf{u}}^{(2)}, \underline{\mathbf{w}}^{(2)}\big) \leq \big(\overline{\mathbf{u}}^{(2)}, \overline{\mathbf{w}}^{(2)}\big) \leq \big(\overline{\mathbf{u}}^{(1)}, \overline{\mathbf{w}}^{(1)}\big),$$

$\overline{\mathbf{u}}^{(2)}$ and $\underline{\mathbf{u}}^{(2)}$ are coupled upper and lower solutions of (1.1). The conclusion of the lemma follows from the induction principle. □

In view of Lemma 1, the pointwise limits

$$\lim_{m\to\infty}\big(\overline{\mathbf{u}}^{(\mathbf{m})}, \overline{\mathbf{w}}^{(\mathbf{m})}\big) = (\overline{\mathbf{u}}, \overline{\mathbf{w}}), \qquad \lim_{m\to\infty}\big(\underline{\mathbf{u}}^{(\mathbf{m})}, \underline{\mathbf{w}}^{(\mathbf{m})}\big) = (\underline{\mathbf{u}}, \underline{\mathbf{w}})$$

exist. Since (2.2) and (2.9), $\overline{D}_i(\overline{u}_i^{(m)}) = D_i(\overline{u}_i^{(m)})$ for $i = 1, 2$. Then by letting $m \to \infty$ in (2.8), we obtain the following system

$$\begin{aligned}
&\partial\overline{u}_1/\partial t - \nabla\cdot\big(D_1(\overline{u}_1)\nabla\overline{u}_1\big) = \overline{u}_1(a_1 - b_{11}\overline{u}_1 - b_{12}\underline{u}_2 - e_1\overline{u}_1\underline{u}_2),\\
&\partial\overline{u}_2/\partial t - \nabla\cdot\big(D_2(\overline{u}_2)\nabla\overline{u}_2\big) = \overline{u}_2\big(a_2 - b_{21}\underline{u}_1 - b_{22}\overline{u}_2 - e_2(\underline{u}_1)_\tau\overline{u}_2\big),\\
&\partial\underline{u}_1/\partial t - \nabla\cdot\big(D_1(\underline{u}_1)\nabla\underline{u}_1\big) = \underline{u}_1(a_1 - b_{11}\underline{u}_1 - b_{12}\overline{u}_2 - e_1\underline{u}_1\overline{u}_2),\\
&\partial\underline{u}_2/\partial t - \nabla\cdot\big(D_2(\underline{u}_2)\nabla\underline{u}_2\big) = \underline{u}_2\big(a_2 - b_{21}\overline{u}_1 - b_{22}\underline{u}_2 - e_2(\overline{u}_1)_\tau\underline{u}_2\big) \quad \text{in } Q,\\
&\partial\overline{u}_i/\partial\nu = \partial\underline{u}_i/\partial\nu = 0 \quad \text{for } i = 1, 2, \text{ on } S,\\
&\overline{u}_i(t,x) = \underline{u}_i(t,x) = \psi_i(t,x) \quad \text{for } i = 1, 2, \text{ in } D_0^{(i)}.
\end{aligned} \tag{2.13}$$

Next we show that the $(\overline{\mathbf{u}}, \underline{\mathbf{u}})$ are the solutions of (2.13) using the standard regularity argument.

Lemma 2. $(\overline{\mathbf{u}}(t,x), \underline{\mathbf{u}}(t,x))$ *is the solution of* (2.13).

Proof. Since (2.8), $\overline{u}_i^{(m)}$ $(i = 1, \ldots, N)$ is a solution of the scalar quasilinear system

$$\begin{aligned}
&\partial u/\partial t - \nabla\cdot\big(a_i D_i(u)\nabla u\big) + \mathbf{b_i}\cdot\big(D_i(u)\nabla u\big) + \beta_i I_i(u)\\
&\quad = F_i\big(\cdot, \overline{u}_i^{(m-1)}, \overline{u}_i^{(m-1)}, \underline{u}_j^{(m-1)}\big) \quad \text{in } D,\\
&\partial u/\partial\nu = 0 \quad \text{on } S, \quad u(t,x) = \psi_i(t,x) \quad \text{in } D_0^{(i)},
\end{aligned}$$

by hypothesis (H_1) and Theorem 7.4 of Chapter V in [8], there is a constant $\alpha > 0$ such that $\overline{u}_i^{(m)} \in C^{1+\alpha/2, 2+\alpha}(\overline{Q})$ for all $m = 1, 2, \ldots$. Furthermore, since the sequence $\overline{u}_i^{(m)}$ is uniformly bounded in $C(\overline{Q})$, it follows from Theorem 7.2 of Chapter V in [8] that there exist positive constants M and δ, independent of m, such that

$$\left|\overline{u}_i^{(m)}\right|_{C^{1+\alpha/2,2+\alpha}(\overline{Q})} \le M, \qquad \left|\nabla \overline{u}_i^{(m)}\right|_{C(\overline{Q})} \le M. \tag{2.14}$$

We now show that the limit $\overline{u}_i$ of $\overline{u}_i^{(m)}$ satisfies the first equation of (2.13) in Q. For each $\overline{u}_i^{(m)}$, we let the operator $\mathcal{L}^{(m)}$ and the function $\mathcal{F}^{(m)}$ be defined by

$$\begin{aligned} \mathcal{L}^{(m)} &= \partial u/\partial t - \nabla \cdot \big(a_i D_i(u)\nabla u\big) + \mathbf{b_i} \cdot \big(D_i(u)\nabla u\big) \\ &\equiv \partial u/\partial t - a_i D_i(u)\nabla^2 u + \big[D_i(u)\nabla a_i + a_i D_i'(u)\nabla u + D_i(u)\mathbf{b_i}\big] \cdot \nabla u, \\ \mathcal{F}^{(m)} &= F_i\big(\cdot, \overline{u}_i^{(m-1)}, \overline{u}_i^{(m-1)}, \underline{u}_j^{(m-1)}\big) - \beta_i I_i\big(u(t,x)\big). \end{aligned} \tag{2.15}$$

In view of (2.11), $\overline{u}_i^{(m)}$ satisfies the linear equation $\mathcal{L}^{(m)} u = \mathcal{F}^m$ in Q. Let Q' be an arbitrary subdomain of Q whose distance from Q has a positive lower bound. Then by (2.15), all the conditions in Theorem 15 of [4] are satisfied in Q'. Hence there is a subsequence $\overline{u}^{(m')}$ such that $\nabla \overline{u}^{(m')}$, $\nabla^2 \overline{u}^{(m')}$, and $\partial \overline{u}^{(m')}/\partial t$ are all uniformly convergent in $\overline{Q}_T'$, and the coefficients of $\mathcal{L}^{(m)}$ converge to the corresponding limits. This proves that the limit $\overline{u}_i$ is in $C^{1,2}(\overline{Q}')$ and satisfies the equation

$$\frac{\partial u}{\partial t} - \nabla \cdot \big(a_i D_i(u)\nabla u\big) + \mathbf{b_i} \cdot \big(D_i(u)\nabla u\big) = f_i\big(\cdot, \overline{u}_i^{(m-1)}, \overline{u}_i^{(m-1)}, \underline{u}_j^{(m-1)}\big) \text{ in } Q'.$$

Since Q' is arbitrary, $\overline{u}_i$ satisfies the first equation of (2.13) in the whole domain Q. Similarly, $\underline{u}_i$ also satisfies the second equation of (2.13). Thus the proof is completed. □

We call $\overline{\mathbf{u}}$, $\underline{\mathbf{u}}$ quasisolutions of (1.1). Moreover, by observing the first and fourth equation of (2.13), we conclude that $(\overline{u}_1, \underline{u}_2)$ is the solution of (1.1). Similarly, $(\underline{u}_1, \overline{u}_2)$ is also the solution of (1.1). It thus leads to the existence theorem from Lemma 2.

Theorem 1. *Let $\tilde{\mathbf{u}}$, $\hat{\mathbf{u}}$ be a pair of coupled upper and lower solutions of* (1.1). *Assume that $\hat{\mathbf{u}} \ge \delta > 0$, then*

(i) $(\overline{u}_1, \underline{u}_2)$ *and* $(\underline{u}_1, \overline{u}_2)$ *are the solutions of* (1.1). *Moreover for all $m \ge 1$*

$$\hat{\mathbf{u}} \le \underline{\mathbf{u}}^{(\mathbf{m})} \le \underline{\mathbf{u}}^{(\mathbf{m+1})} \le \underline{\mathbf{u}} \le \overline{\mathbf{u}} \le \overline{\mathbf{u}}^{(\mathbf{m+1})} \le \overline{\mathbf{u}}^{(\mathbf{m})} \le \tilde{\mathbf{u}} \ \text{ in } \overline{D}. \tag{2.16}$$

(ii) *If $\underline{\mathbf{u}} = \overline{\mathbf{u}} (\equiv \mathbf{u}^*)$, then $\mathbf{u}^*$ is the unique solution in Λ.*

Next we discuss the asymptotic behaviour of the problem (1.1) by using the method as in [15, 16, 17, 18]. Assume that the coupled upper and lower solutions are constant vectors $\tilde{\mathbf{c}} \equiv (\tilde{c}_1, \tilde{c}_2)$, $\hat{\mathbf{c}} \equiv (\hat{c}_1, \hat{c}_2)$. By using $\underline{\mathbf{c}}^{(0)} = \hat{\mathbf{c}}$ and $\overline{\mathbf{c}}^{(0)} = \tilde{\mathbf{c}}$ as the initial iteration, after the nonlinear iteration process (2.8), we can construct constant sequences $\{\underline{\mathbf{c}}^{(m)}, \underline{\mathbf{c}}^{(m)}\}$ and $\{\overline{\mathbf{c}}^{(m)}, \overline{\mathbf{c}}^{(m)}\}$. We set the pointwise limits

$$\lim_{m\to\infty} \big(\overline{c}_1^{(m)}, \overline{c}_2^{(m)}\big) = (\overline{c}_1, \overline{c}_2), \qquad \lim_{m\to\infty} \big(\underline{c}_1^{(m)}, \underline{c}_2^{(m)}\big) = (\underline{c}_1, \underline{c}_2).$$

Such as in (2.16), $(\overline{c}_1, \overline{c}_2)$ and $(\underline{c}_1, \underline{c}_2)$ satisfy the following relation

$$\begin{aligned}&\overline{c}_1(a_1 - b_{11}\overline{c}_1 - b_{12}\underline{c}_2 - e_1\overline{c}_1\underline{c}_2) = 0 = \underline{c}_1(a_1 - b_{11}\underline{c}_1 - b_{12}\overline{c}_2 - e_1\underline{c}_1\overline{c}_2),\\ &\overline{c}_2(a_2 - b_{21}\underline{c}_1 - b_{22}\overline{c}_2 - e_2\underline{c}_1\overline{c}_2) = 0 = \underline{c}_2(a_2 - b_{21}\overline{c}_1 - b_{22}\underline{c}_2 - e_2\overline{c}_1\underline{c}_2).\end{aligned} \tag{2.17}$$

After the similar argument as in the proof of Lemma 2, we have the following asymptotic property.

Theorem 2. *Let* $\tilde{\mathbf{c}}$, $\hat{\mathbf{c}}$ *be a pair of coupled constant upper and lower solutions of* (1.1). *Assume that* $\hat{\mathbf{c}} \geq \delta > 0$, *then the quasisolutions* $(\overline{c}_1, \overline{c}_2)$ *and* $(\underline{c}_1, \underline{c}_2)$ *satisfy* (2.17) *and*

$$\hat{\mathbf{c}} \leq \underline{\mathbf{c}}^{(\mathbf{m})} \leq \underline{\mathbf{c}}^{(\mathbf{m+1})} \leq \underline{\mathbf{c}} \leq \overline{\mathbf{c}} \leq \overline{\mathbf{c}}^{(\mathbf{m+1})} \leq \overline{\mathbf{c}}^{(\mathbf{m})} \leq \tilde{\mathbf{c}} \quad \text{in } \overline{Q},\ m = 1, 2, \ldots. \tag{2.18}$$

If $(\overline{c}_1, \overline{c}_2) = (\underline{c}_1, \underline{c}_2) \equiv (c_1^*, c_2^*)$, *then for any* $\hat{c}_i \leq \psi_i(t,x) \leq \tilde{c}_i$ *in* D_i, *the unique solution* $(u_1(t,x), u_2(t,x))$ *possesses the convergence property is the unique solution*

$$\lim_{t\to\infty} \big(u_1(t,x), u_2(t,x)\big) = (c_1^*, c_2^*) \quad (x \in \bar{\Omega}).$$

3 Existence and Asymptotic Behavior of Solutions

First we show the existence by seeking the coupled upper and lower solutions of (1.1). It is easy to verify that if $(\tilde{u}_1, \tilde{u}_2)$ and $(\hat{u}_1, \hat{u}_2)$ satisfy $(\tilde{u}_1, \tilde{u}_2) \geq (\hat{u}_1, \hat{u}_2)$ and the following inequalities

$$\begin{aligned}&\partial\tilde{u}_1/\partial t - \nabla\cdot\big(D_1(\tilde{u}_1)\nabla\tilde{u}_1\big) \geq \tilde{u}_1(a_1 - b_{11}\tilde{u}_1 - b_{12}\hat{u}_2 - e_1\tilde{u}_1\hat{u}_2),\\ &\partial\tilde{u}_2/\partial t - \nabla\cdot\big(D_2(\tilde{u}_2)\nabla\tilde{u}_2\big) \geq \tilde{u}_2\big(a_2 - b_{21}\hat{u}_1 - b_{22}\tilde{u}_2 - e_2(\hat{u}_1)_\tau\tilde{u}_2\big),\\ &\partial\hat{u}_1/\partial t - \nabla\cdot\big(D_1(\hat{u}_1)\nabla\hat{u}_1\big) \leq \hat{u}_1(a_1 - b_{11}\hat{u}_1 - b_{12}\tilde{u}_2 - e_1\hat{u}_1\tilde{u}_2),\\ &\partial\hat{u}_2/\partial t - \nabla\cdot\big(D_2(\hat{u}_2)\nabla\hat{u}_2\big) \leq \hat{u}_2\big(a_2 - b_{21}\tilde{u}_1 - b_{22}\hat{u}_2 - e_2(\tilde{u}_1)_\tau\hat{u}_2\big) \quad \text{in } D,\\ &\partial\hat{u}_i/\partial\nu \leq 0 \leq \partial\tilde{u}_i/\partial\nu \quad \text{for } i = 1, 2, \text{ on } S,\\ &\hat{u}_i(t,x) \leq \psi_i(t,x) \leq \tilde{u}_i(t,x) \quad \text{for } i = 1, 2, \text{ in } D_i.\end{aligned} \tag{3.1}$$

then the pair $(\tilde{u}_1, \tilde{u}_2)$, $(\hat{u}_1, \hat{u}_2)$ are coupled upper and lower solutions of (1.1). To guarantee (3.1), we seek such a constant pair in the form $(\tilde{c}_1, \tilde{c}_2) = (M_1, M_2)$, $(\hat{c}_1, \hat{c}_2) = (\delta_1, \delta_2)$, where for each $i = 1, 2$, M_i are positive constants and δ_i are some sufficiently small constants. Thus (3.1) is satisfied if

$$\begin{aligned}\tilde{c}_1(a_1 - b_{11}\tilde{c}_1 - b_{12}\hat{c}_2 - e_1\tilde{c}_1\hat{c}_2) \leq 0 \leq \hat{c}_1(a_1 - b_{11}\hat{c}_1 - b_{12}\tilde{c}_2 - e_1\hat{c}_1\tilde{c}_2),\\ \tilde{c}_2(a_2 - b_{21}\hat{c}_1 - b_{22}\tilde{c}_2 - e_2\hat{c}_1\tilde{c}_2) \leq 0 \leq \hat{c}_2(a_2 - b_{21}\tilde{c}_1 - b_{22}\hat{c}_2 - e_2\tilde{c}_1\hat{c}_2),\\ \psi_i(t,x) \leq \tilde{c}_i \quad \text{for } i = 1, 2, \text{ in } D_0^{(i)}.\end{aligned} \tag{3.2}$$

If we set $M_1 = a_1/b_{11} < a_2/b_{21}$, $M_2 = a_2/b_{22} < a_1/b_{12}$, then (3.2) is satisfied. Therefore we conclude the existence results.

Theorem 3. *Suppose there exists a positive constant* δ *such that the initial functions* $\delta \leq \psi_i(x,t) < a_i/b_{ii}$ $(i = 1, 2)$ *in* $D_0^{(i)}$. *If*

$$b_{12}/b_{22} < a_1/a_2 < b_{11}/b_{21}, \tag{3.3}$$

then the system (1.1) *admits at least one positive solution.*

In order to investigate the asymptotic behaviour of the coupled system (1.1), we consider the scalar logistic parabolic equation

$$\begin{cases} \partial u_1/\partial t - \nabla\cdot\big(d_1(u_1/n_1)^m\nabla u_1\big) = u_1(a_1 - b_{11}u_1) \quad (t>0,\ x\in\Omega),\\ \partial u_1/\partial\nu = 0 \quad (t>0,\ x\in\partial\Omega),\\ u_1(t,x) = \psi_1(t,x) \quad \big(t\in[-\tau,0],\ x\in\Omega\big). \end{cases} \tag{3.4}$$

Lemma 3. *There exists the unique solution* $u_1(x,t)$ *of* (3.4)*, which satisfies*

$$\lim_{t\to\infty} u_1(x,t) = a_1/b_{11}.$$

Proof. It is obvious that

$$\tilde{c}_1 = \max\Big\{a_1/b_{11}, \max_{(x,t)\in D_1}\psi_1(t,x)\Big\}, \qquad \hat{c}_1 = \min_{(x,t)\in D_1}\psi_1(t,x)$$

are upper and lower solutions of (3.4). Thus the existence of $u_1(x,t)$ is ensured by Theorem 1. Applying Theorem 2, (2.17) is reduced to

$$\overline{c}_1(a_1 - b_{11}\overline{c}_1) = 0 = \underline{c}_1(a_1 - b_{11}\underline{c}_1). \tag{3.5}$$

Since (2.16), $0 < \underline{c}_1 < \overline{c}_1$ holds. Thus it follows from (3.5) that $\underline{c}_1 = \overline{c}_1 = a_1/b_{11}$, which proves the theorem by using Theorem 2. □

We now give the global attractivity of the system (1.1) in the following lemma.

Lemma 4. *For any given initial function* $\psi_i(x,t)\ge\delta$ *in* $D_0^{(i)}$ *for* $i=1,2$*, if* (3.3) *holds, then the solution* $(u_1(x,t),u_2(x,t))$ *of the system* (1.1) *satisfies*

$$\lim_{t\to\infty} u_1(x,t) \le \frac{a_1}{b_{11}}, \qquad \lim_{t\to\infty} u_2(x,t) \le \frac{a_2}{b_{22}} \quad \textit{in } Q. \tag{3.6}$$

Proof. We essentially use the positive lemma of Lemma 2.1 in [20]. Considering the scalar quasilinear parabolic problem as the following

$$\begin{cases} \partial u_1/\partial t - \nabla\cdot\big((u_1/n_1)^m\nabla u_1\big) = u_1(a_1 - b_{11}u_1 - b_{12}u_2)\\ \qquad\qquad\qquad\qquad\qquad\qquad \le u_1(a_1 - b_{11}u_1) \quad \text{in } Q,\\ \partial u_1/\partial\nu = 0 \quad \text{on } S,\\ u_1(t,x) = \psi_1(t,x) \quad \text{in } D_0^{(1)}, \end{cases}$$

in view of the positive lemma that $u_1(t,x)\le U_1(t,x)$, where $U_1(t,x)$ is the unique solution of (3.4). Letting $t\to\infty$, $\lim_{t\to\infty}u_1(x,t)\le\lim_{t\to\infty}U_1(x,t)\le a_1/b_{11}$. Similarly, $\lim_{t\to\infty}u_2(x,t)\le a_2/b_{22}$. Thus the proof is completed. □

From Lemma 4, for any initial functions, the solution of (1.1) will go into the attractor $[0,a_1/b_{11}]\times[0,a_2/b_{22}]$. Therefore in this case Theorem 3 is valid.

Corollary 1. If (3.3) holds, then the system (1.1) admits at least one positive solution. Moreover, (3.6) holds.

Now we study the global stability of the uniform equilibrium for (1.1). After some algebra calculations similar as in the argument in [12], we get the sufficient conditions such that system (1.1) has the positive uniform equilibrium, moreover the positive uniform equilibrium is locally stable.

Lemma 5. *If*

$$b_{12}/b_{22} < a_1/a_2 < b_{11}/b_{21}, \quad b_{12}/b_{22} < e_1/e_2 < b_{11}/b_{21}, \tag{3.7}$$

then (1.1) *admits the unique positive uniform equilibrium* $E^*\colon (N_1^*, N_2^*)$, *where* N_1^*, N_2^* *are the positive constants depending on the coefficients* a_i, b_{ij}, e_i.

Lemma 6. *If* (3.7) *holds, then the positive uniform equilibrium* $E^*\colon (N_1^*, N_2^*)$ *of* (1.1) *is locally stable.*

Proof. First, for sake of simplicity, we denote

$$\mathbf{f}(\mathbf{u}) = \begin{pmatrix} f_1(u_1, u_2) \\ f_2(u_1, u_2, (u_1)_\tau) \end{pmatrix}.$$

The linearization of (1.1) around the state E^* gives

$$\partial \mathbf{u}/\partial t - D\Delta \mathbf{u}^{m+1} = \mathbf{f_u}(E^*)\mathbf{u}, \tag{3.8}$$

where $D = \mathrm{diag}(d_1/(m+1)n_1^m, d_2/(m+1)n_2^m)$, and

$$\mathbf{f_u}(E^*) = \begin{pmatrix} -N_1^*(b_{11} + e_1 N_2^*) & -N_1^*(b_{12} + e_1 N_1^*) \\ -N_2^*(b_{21} + e_2 N_2^* e^{-\lambda\tau}) & -N_2^*(b_{22} + e_2 N_1^*) \end{pmatrix} \triangleq \begin{pmatrix} \mathbf{f_{u11}} & \mathbf{f_{u12}} \\ \mathbf{f_{u21}} & \mathbf{f_{u22}} \end{pmatrix}.$$

Setting $0 = \mu_1 < \mu_2 < \cdots \to \infty$ be the eigenvalues of $-\Delta \mathbf{u}^{m+1}$ on Ω under no-flux boundary conditions, direct calculation shows that the characteristic polynomial of (3.8) is given by

$$\psi_i(\lambda) = \lambda^2 - B_i \lambda + C_i,$$

where

$$\begin{aligned} B_i &= -N_1^*(b_{11} + e_1 N_2^*) - N_2^*(b_{22} + e_2 N_1^*) - d_1\mu_i - d_2\mu_i, \\ C_i &= \big(N_1^*(b_{11} + e_1 N_2^*) + d_1\mu_i\big)\big(N_2^*(b_{22} + e_2 N_1^*) + d_2\mu_i\big) \\ &\quad - N_1^* N_2^*(b_{12} + e_1 N_1^*)\big(b_{21} + e_2 N_2^* e^{-\lambda\tau}\big). \end{aligned}$$

Recalling condition (3.7), it is easy to verify that B_i and C_i are negative. Thus, for each $i \geq 1$, the two roots $\lambda_{i,1}$, $\lambda_{i,2}$ of $\psi_i(\lambda) = 0$ all have negative real parts, and this concludes the proof. □

Based on the local stability of the equilibrium of (1.1), we can have the global stability of the equilibrium by using of Theorem 2.

Theorem 4. *Suppose there exists a positive constant* δ *such that the initial functions* $\delta \leq \psi_i(x,t)$ $(i = 1, 2)$ *in* $D_0^{(i)}$. *If* (3.7) *holds, then the positive* $E^*\colon (N_1^*, N_2^*)$ *is asymptotically stable.*

Proof. In view of Lemma 4, we have the following estimates

$$\lim_{t\to\infty} u_1(x,t) \le a_1/b_{11}, \quad \lim_{t\to\infty} u_2(x,t) \le a_2/b_{22}.$$

For any given ε_1, $\varepsilon_2 > 0$, there exists $t_0 > 0$ such that $u_1(x,t) \le a_1/b_{11} + \varepsilon_1$, $u_2(x,t) \le a_2/b_{22} + \varepsilon_2$. As the asymptotic behaviour is concerned, it suffices to consider the case $t \ge t_0$. Hence the globally asymptotic behaviour of system (1.1) for any initial function is equivalent to $0 < \eta_i(x,t) \le a_i/b_{ii} + \varepsilon_i$ $(i = 1,2)$. In order to apply the conclusions of Theorem 2, we seek the coupled upper and lower solutions $\tilde{\mathbf{c}}$ and $\hat{\mathbf{c}}$.

We set $\tilde{c}_1 = a_1/b_{11} + \varepsilon_1$, $\tilde{c}_2 = a_2/b_{22} + \varepsilon_2$, $\hat{c}_1 = \delta_1$, $\hat{c}_2 = \delta_2$, where

$$\begin{aligned}
&\varepsilon_1 = a_2 b_{11} - a_1 b_{21}/2b_{11}b_{21}, \quad \varepsilon_2 = a_1 b_{22} - a_2 b_{12}/2b_{12}b_{22},\\
&\delta_1 \le b_{12}(a_1 b_{22} - a_2 b_{12})/\big[2b_{11}b_{12}b_{22} + e_1(a_1 b_{22} + a_2 b_{12})\big],\\
&\delta_2 \le b_{21}(a_2 b_{11} - a_1 b_{21})/\big[2b_{11}b_{21}b_{22} + e_2(a_1 b_{21} + a_2 b_{11})\big].
\end{aligned}$$

It follows from (3.7) that ε_1, ε_2, δ_1, δ_2 are positive. After some algebra calculations, $\tilde{\mathbf{c}}$, $\hat{\mathbf{c}}$ satisfy (3.2), which ensures $\tilde{\mathbf{c}}$ and $\hat{\mathbf{c}}$ are coupled upper and lower solutions of the system (1.1). From (2.18), the quasisolutions $\overline{\mathbf{c}}$ and $\underline{\mathbf{c}}$ satisfy $\overline{\mathbf{c}} \ge \underline{\mathbf{c}} > 0$. (2.17) is reduced to

$$\begin{cases} a_1 - b_{11}\overline{c}_1 - b_{12}\underline{c}_2 - e_1\overline{c}_1\underline{c}_2 = 0 = a_1 - b_{11}\underline{c}_1 - b_{12}\overline{c}_2 - e_1\underline{c}_1\overline{c}_2,\\ a_2 - b_{21}\underline{c}_1 - b_{22}\overline{c}_2 - e_2\underline{c}_1\overline{c}_2 = 0 = a_2 - b_{21}\overline{c}_1 - b_{22}\underline{c}_2 - e_2\overline{c}_1\underline{c}_2. \end{cases} \tag{3.9}$$

The relation (3.9) induces the following

$$(b_{21}e_1 - b_{11}e_2)(\overline{c}_1 - \underline{c}_1) + (b_{12}e_2 - b_{22}e_1)(\overline{c}_2 - \underline{c}_2) = 0. \tag{3.10}$$

Combining (3.7) and (3.10), we have $\overline{c}_1 = \underline{c}_1 = N_1^*$, $\overline{c}_2 = \underline{c}_2 = N_2^*$. By Theorem 2, we conclude $\lim_{t\to\infty}(u_1(x,t), u_2(x,t)) = (N_1^*, N_2^*)$. □

4 Discussions

The main method in Section 2 is the upper and lower solutions and its monotone iteration. The technique is used in many papers for analysis of coupled parabolic systems, see, e.g. [5, 6, 7, 10, 14]. The virtue of the technique is by which the results of existence and stability is extended from the scalar equation to the coupled system. In particular, Pao and Ruan [20] have developed the method to deal with quasilinear parabolic system. Because iterative sequences in Section 2 are independent of the time, the delayed reaction terms of (1.1) can be obtained in analytical form. Our conclusions of Theorems 1, 2 are the extensions from the quasi-monotone nondecreasing reaction term of [20] to mixed quasi-monotone nondecreasing reaction term. By comparing the results between the quasilinear parabolic system and semilinear parabolic system [26], the conditions for the positive uniform equilibrium being asymptotic stable are coincident. It means that the theory of existence and stability in Theorems 1, 2 is usefulness and applicable to the general Lotka–Volterra type model.

All of the previous phytoplankton models, such as [2, 3, 9, 11, 12, 23, 24, 25, 26], did not take density-dependent diffusion effects into account. In fact, plankton can move around subject to diffusion. Thus, it is more realistic to introduce the density-dependent diffusion of the plankton into the system. The concern of the density-dependent diffusion is reasonable in animal disperse model (see [22] for a review). The biological implications of Theorems 3 and 4 are that if the ratios of the intra-competition to the inter-competition belong to some parameter domain, the long term behaviours of the species tend to the positive uniform equilibrium. The two competitive species are coexistent. The results also have applicability to 3 species Lotka–Volterra model.

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