ISSN: 1392-6292/ eISSN: 1648-3510

MATHEMATICAL MODELLING and ANALYSIS

2025
Volume 30
Issue 1
Pages 52–73
https://doi.org/10.3846/mma.2025.20268

Solving class of mixed nonlinear multi-term fractional Volterra-Fredholm integro-differential equations by new development of HAM

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Article History:

- received October 31, 2023
- revised September 6, 2024
- accepted September 17, 2024

Abstract. This work implements the standard Homotopy Analysis Method (HAM) developed by Professor Shijun Liao (1992), and a new development of the HAM (called ND-HAM) improved by Z.K. Eshkuvatov (2022) in solving mixed nonlinear multi-term fractional derivative of different orders of Volterra-Fredholm Integro-differential equations (FracVF-IDEs). Other than that, the existance and uniqueness of solution as well as the norm convergence with respect to ND-HAM, were proven in a Hilbert space. In addition, three numerical examples (including multi-term fractional IDEs) are presented and compared with the HAM, modified HAM and "Generalized block pulse operational differentiation matrices method" developed in previous works by illustrating the accuracy as well as validity with respect to ND-HAM. Empirical investigations reveal that ND-HAM and the modified HAM yields the same results when control parameter $\hbar$ is chosen as $\hbar = -1$ and is comparable to the standard HAM. The findings discovered that the ND-HAM is highly convenient, effective, as well as in line with theoretical results.

Keywords: homotopy analysis method (HAM); new development of HAM (ND-HAM); integro-differential equation (IDEs); Caputo fractional derivative; convergence.

AMS Subject Classification: 65R20; 45E05.

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1 Introduction

IDEs are widely recognized, resulting from the mathematical modelling of scientific phenomena. Linear and nonlinear IDEs can be used to model a variety of nonlinear phenomena. Numerous numerical method series have been established throughout the years to solve IDEs, for instance, the successive approximation method [12, 28], the Adomian decomposition method [3, 20], the differential transform method [2, 7], the Tau method [18], the wavelet method [5], least squares-support vector regression method [22] as well as fixed point method [21].

Numerous applications with regard to fractional differential equations (FDEs) in the fields of physics, engineering, and medical sciences have sparked a great deal of interest in them in recent years. FDEs are useful for describing many significant phenomena in the electromagnetism, materials science, acoustics, electrochemistry, and viscoelasticity fields (see Milici et al. [19], West et al. [31] and Podlubny [23]). Using cutting-edge semi-analytical techniques, several exact solutions to linear FDEs have been discovered. Santra and Mohapatra [25, 26] proposed the classical $L1$ scheme for solving time fractional initial boundary value problem of mixed parabolic - elliptic type with a mild singularity at the initial time $t = 0$ and time fractional partial integro-differential equation of Volterra type respectively. Unfortunately, only a few methods produce exact solutions to the nonlinear FDEs.

Among the most well-known non-perturbative available solution techniques is known as the HAM, which was initially established by Liao [15, 16, 17]. It is deemed a powerful semi-analytical method for solving nonlinear and linear differential and integral equations. There exist many HAM as well as homotopy perturbation method (HPM) implementations into various problems, for instance, the nonlinear Riccati differential equation with fractional order [4], the fractional KdV-Burgers-Kuramoto equation [29], fractional wave equations [14], the MHD-flow of an Oldroyd 8-constant fluid problems [13], numerous nonlinear and linear fractional IDEs [1,9,11], Sobelov method for singular integral equations [27], mixed Volterra-Fredholm integral equations [30] and others.

This work considers mixed nonlinear VF-IDEs of multi-term fractional orders with respect to those given by

$$\big({}^{c}\mathfrak{D}^{\ell_p}_{0^+} + \sum_{j=0}^{p-1} \vartheta_j \, {}^{c}\mathfrak{D}^{\ell_j}_{0^+}\big) u(t) = \Upsilon(t) + \omega \int_0^t \int_0^T \Lambda(x,s) \eth(u(s)) \, ds \, dx, \tag{1.1}$$

having initial conditions

$$u^{(k)}(0) = \theta_k \quad \text{for} \quad k = 0, \dots, p-1, \tag{1.2}$$

or boundary conditions

$$u^{(k)}(0) = \theta_k \quad \text{for} \quad k = 0, \dots, p-2 \qquad \text{and} \qquad u(T) = B, \tag{1.3}$$

in which $t \in \mho = [0,T]$; $\Lambda\colon \mho \times \mho \to \mathbb{R}$ as well as $\Upsilon\colon \mho \to \mathbb{R}$ are known functions, $\eth\colon C(\mho, \mathbb{R}) \to \mathbb{R}$ denotes a nonlinear function, ϑ_j, ω, B, θ_k, $p \geq 2$ resembles constants integer; $j-1 < \ell_j \leq j$ for $j = 1, 2, \dots, p$, while ${}^{c}\mathfrak{D}^{\ell_j}_{0^+}$ refers to the Caputo fractional derivative having order ℓ_j. Moreover, ℓ_p is known as the *order* of (1.1) while $C([0,T], \mathbb{R})$ expresses the Banach space of continuous real valued functions on $[0,T]$ to $\mathbb{R}$, enabled by the norm $\|u\| = \sup\limits_{t\in[0,T]} |u(t)|$.

This work describes an application of *New Development of the Homotopy Analysis Method* (ND-HAM) proposed by Eshkuvatov [10] in 2021 for solving mixed nonlinear VF-IDEs involving multi-term fractional order. Section 2 recalls several significant fundamental concepts with respect to fractional calculus, while Section 3 describes the ND-HAM, studies its convergence, and proves the uniqueness of the solution for (1.1)–(1.3). Meanwhile, Section 4 provides two examples in illustrating the performance with regard to the suggested method and for comparison purposes with respect to other methods. Lastly, the conclusion of the paper is given in Section 5.

2 Preliminaries

This section demonstrates the fundamentals with regard to fractional calculus, commonly established as the derivatives and integrals theories of arbitrary order. It generalizes the notions pertaining to the integer-order differentiation and n-fold integration. Note that many available books are available (for example, West et al. [31] as well as Podlubny [23]) studying fractional calculus and numerous definitions with respect to fractional differentiation and integration. This includes Caputo's definition, Riemann-Liouville's definition, as well as the generalized function approach.

Definition 1. Let $x > 0$, $\mu \in \mathbb{R}$ and $m \in \mathbb{N}$. A real function $\wp(x)$ is in the space C_μ provided that there exists a real number $p > \mu$ such that $\wp(x) = x^p \wp_1(x)$, in which $\wp_1(x) \in C[0,\infty)$. Moreover, it is also known to be in the C_μ^m for $\wp^{(m)} \in C_\mu$ space. Evidently, $C_\mu \subset C_\nu$ provided that $\mu \geq \nu$.

Definition 2. Let $\wp \in C_\mu$ for various $\mu > -1$. Here, the Riemann-Liouville fractional integral operator having order $\theta \in \mathbb{R}^+$ of $\wp$ is expressed by

$$J^\theta_{a+}\wp(t) = \begin{cases} \wp(t), & \text{if } \theta = 0, \\ \dfrac{1}{\Gamma(\theta)} \displaystyle\int_a^t (t-\varsigma)^{\theta-1}\wp(\varsigma)\, d\varsigma, & \text{if } \theta > 0\ , \end{cases}$$

in which Γ refers to the Euler's Gamma function defined by

$$\Gamma(\theta) = \int_0^{+\infty} t^{\theta-1}e^{-t}\, dt,\ \Gamma(n) = (n-1)!.$$

Definition 3. Assume $\wp^{(n)} \in L^1[a,b]$ as well as $\theta \in \mathbb{R}^+$ provided that $n-1 < \theta \leq n$ for some $n \in \mathbb{N}$. The Riemann-Liouville fractional derivative of order θ is expressed by

$$^{RL}\mathfrak{D}^\theta_{a+}\wp(t) =$$

$$\begin{cases} \wp(t), & \text{if } \theta = 0, \\ \wp^{(n)}(t), & \text{if } \theta = n, \\ D^n_{a^+}J^{n-\theta}_{a^+}\wp(t) = \dfrac{d^n}{dt^n}\left(\dfrac{1}{\Gamma(n-\theta)}\displaystyle\int_a^t (t-\varsigma)^{n-\theta-1}\wp(\varsigma)d\varsigma\right), & \text{if } n-1 < \theta < n. \end{cases}$$

Definition 4. The Caputo fractional derivative having order $\theta \in \mathbb{R}^+$ with respect to a function $\wp$ is expressed by

$$
{}^{c}\mathfrak{D}^{\theta}_{a+}\wp(t) = \begin{cases} \wp(t), & \text{if } \theta = 0, \\ \wp^{(n)}(t), & \text{if } \theta = n, \\ J^{n-\theta}_{a+}\left(\dfrac{d^n}{dt^n}\wp(t)\right) = \dfrac{1}{\Gamma(n-\theta)}\displaystyle\int_a^t (t-\varsigma)^{n-\theta-1}\wp^{(n)}(\varsigma)\,d\varsigma & \text{if } n-1<\theta<n, \end{cases}
$$

in which $\wp^{(n)} \in L^1[a,b]$ and $n-1<\theta\le n$ for some $n \in \mathbb{N}$.

Following Caputo's derivative definition, we now obtain

$$
\begin{cases} {}^{c}\mathfrak{D}^{\theta}_{a+} c = 0, & \text{if } c \text{ is a constant}, \\ {}^{c}\mathfrak{D}^{\theta}_{a+}(t-a)^{\alpha} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-\theta)}(t-a)^{\alpha-\theta}, & \text{if } \alpha \ge \theta, \\ {}^{c}\mathfrak{D}^{\theta}_{a+}(t-a)^{\alpha} = 0, & \text{if } \alpha < \theta. \end{cases}
$$

Also, we can see that Caputo's fractional derivative is a linear operation, which can be expressed as

$$
{}^{c}\mathfrak{D}^{\theta}_{a+}\big(\omega\wp(t)+\mu g(t)\big) = \omega\,{}^{c}\mathfrak{D}^{\theta}_{a+}\wp(t) + \mu\,{}^{c}\mathfrak{D}^{\theta}_{a+}g(t),
$$

in which ω as well as μ are constants.

Leibniz rule. From elementary calculus, we know that the product's derivative with respect to the two functions $\wp(t)$ as well as $g(t)$ is expressed by

$$
\mathfrak{D}^{N}[\wp(t)\,g(t)] = \sum_{n=0}^{N}\binom{N}{n}\mathfrak{D}^{N-n}\wp(t)\,D^{n}g(t).
$$

A reasonable generalization of this result to fractional derivatives is

$$
{}^{c}\mathfrak{D}^{\theta}_{a+}[\wp(t)\,g(t)] = \sum_{k=0}^{\infty}\binom{\theta}{k}\,{}^{c}\mathfrak{D}^{\theta-k}_{a+}\wp(t)\,\mathfrak{D}^{k}g(t), \quad \theta \in \mathbb{R}^+,
$$

provided that $\wp(t)$ is continuous in $[a,t]$ while $g(t)$ possesses $n+1$ continuous derivatives in $[a,t]$.

Remark 1. Let $\theta>0$, $\ell>0$, and $\wp \in L^1[a,b]$. Following from here, we have:

$$
J^{\theta}_{a+}J^{\ell}_{a+}\wp(t) = J^{\ell}_{a+}J^{\theta}_{a+}\wp(t) = J^{\theta+\ell}_{a+}\wp(t), \tag{2.1}
$$

$$
{}^{c}\mathfrak{D}^{\theta}_{a+}\left[J^{\theta}_{a+}\wp(t)\right] = \wp(t), \tag{2.2}
$$

$$
J^{\theta}_{a+}\left[{}^{c}\mathfrak{D}^{\theta}_{a+}\wp(t)\right] = \wp(t) - \sum_{k=0}^{n-1}\frac{\wp^{(k)}(a)}{k!}(t-a)^k \quad \text{for } n-1<\theta\le n. \tag{2.3}
$$

Also, the fractional integral acts on a power function according to the following formula:

$$
J^{\theta}_{a+}(t-a)^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\theta+\mu+1)}(t-a)^{\theta+\mu}, \tag{2.4}
$$

where $\mu>-1$.

3 Basic idea of the standard HAM and ND-HAM

Assume a general nonlinear VF-IDEs with respect to the fractional order given by

$$N[u(t)] = \Upsilon(t), \tag{3.1}$$

for $t \in \mho$. Liao [15] established the zeroth-order deformation equation with respect to the HAM in the form given by

$$(1-q)\,\pounds\big[\Pi(t;q) - u_0(t)\big] = q\hbar H(t)\big[N[\Pi(t;q)] - \Upsilon(t)\big], \tag{3.2}$$

in which $\pounds$ is a linear differential operator, $q \in [0,1]$ is an embedding parameter, $H(t)$ resembles an auxiliary function, $\hbar \neq 0$ denotes an auxiliary parameter, $u_0(t)$ refers to an initial guess with respect to the solution $u(t)$ that satisfies the boundary or initial conditions pertaining to the equation, while $\Pi(t;q)$ expresses an unknown function to be determined, relying on the variables q as well as t, which satisfies equations given by

$$\Pi^{(i)}(t;0) = u_0^{(i)}(t),$$

for $i = 0,1,2,\ldots$. Provided that the parameter q rises from 0 to 1, the homotopy solution $\Pi(t;q)$ ranges from $u_0(t)$ to $u(t)$. By employing the parameter q as a dummy variable, the function $\Pi(t;q)$ may be expanded as the Taylor series given by

$$\Pi(t;q) = \sum_{m=0}^{+\infty} u_m(t)q^m, \quad u_m(t) = \frac{1}{m!}\left.\frac{\partial^m \Pi(t;q)}{\partial q^m}\right|_{q=0}.$$

Let the auxiliary parameter $\hbar$ be chosen to assure the series given above converges when $q = 1$. Then, the solution $u(t)$ may now be expressed as

$$u(t) = \sum_{m=0}^{+\infty} u_m(t).$$

Consequently, approximate solutions of (3.1) can be obtained as

$$u(t) \approx \sum_{m=0}^{n} u_m(t). \tag{3.3}$$

Standard procedure (High-order deformation equation). To keep the following discussion concise, we use the notation

$$\overline{u}_n = \big\{u_0(t), u_1(t), u_2(t), \ldots, u_n(t)\big\}.$$

Upon differentiating the zeroth-order deformation (3.2) m times concerning the embedding parameter q, dividing by $m!$, as well as setting $q = 0$, we have the so-called *mth-order deformation equation* written as

$$\pounds[u_m(t) - \chi_m u_{m-1}(t)] = \hbar H(t)\Re_m(\overline{u}_{m-1}(t)), \tag{3.4}$$

in which

$$\Re_m(\overline{u}_{m-1}(t))=\frac{1}{(m-1)!}\left.\frac{\partial^{m-1}\left[N[\varPi(t;q)]-\varUpsilon(t)\right]}{\partial q^{m-1}}\right|_{q=0},\ \chi_m=\begin{cases}0 & \text{if } m\le 1,\\ 1 & \text{if } m>1.\end{cases} \tag{3.5}$$

The solution to the approximation method defined by (3.3) and (3.4) is called *standard HAM.*

New procedure: In order to **derive the ND-HAM**, we consider that the function $\varUpsilon(t)$ in Equation (3.1) can be split as the sum of $n+1$ terms:

$$\varUpsilon(t)=s_0(t)+s_1(t)+\ldots+s_n(t). \tag{3.6}$$

Expanding $\varUpsilon(t)$ into powers with regards to the embedding parameter q, yields

$$g(t;q)=s_0(t)+s_1(t)+s_2(t)(-q\hbar)+\ldots+s_n(t)(-q\hbar)^{n-1},$$

where $\hbar$ is the control parameter of (3.2). For ND-HAM, we rewrite (3.4) as

$$\pounds[u_0(t)] = s_0(t), \tag{3.7}$$

$$\pounds[u_m(t)-\chi_m u_{m-1}(t)] = \hbar H(t)\Re_m(\overline{u}_{m-1}(t)), \tag{3.8}$$

in which χ_m is defined as in (3.4) and

$$\Re_m(\overline{u}_{m-1}(t)) = \frac{1}{(m-1)!}\left.\frac{\partial^{m-1}\left[N[\varPi(t;q)]-g(t;q)\right]}{\partial q^{m-1}}\right|_{q=0}. \tag{3.9}$$

The ND-HAM gives us great freedom in choosing the function $s_0(t)$ depending on the given function $\varUpsilon(t)$ in (3.6). In some cases, a suitable choice of the function $s_0(t)$ yields directly the exact solution $u(t)$ with respect to the problem and thus (3.8) resulting in zero for the next iterations, i.e., $u_i(t)=0$ for $i=1,2,\ldots$. On the other hand, if (3.7) does not give us the exact solution. It will provide an adequate initial guess satisfying the initial or boundary conditions. Also, it should be noted that the residual term (3.9) of the ND-HAM requires much simpler computations than the residual term (3.5) of the standard HAM.

4 Primary findings. Application of the HAM and ND-HAM

To solve mixed nonlinear VF-IDEs of the form (1.1)–(1.3) using the HAM and the ND-HAM, we establish the nonlinear operator given below

$$N\left[\varPi(t;q)\right]=\left({}^{c}\mathfrak{D}^{\ell_p}_{0^+}+\sum_{j=0}^{p-1}\vartheta_j\,{}^{c}\mathfrak{D}^{\ell_j}_{0^+}\right)\varPi(t;q)-\omega\int_0^t\int_0^T \varLambda(x,s)\eth(\varPi(s;q))\,ds\,dx,$$

where $\varPi(t;q)$ refers to an unknown function that needs to be determined. Therefore, we may express the problem (1.1)–(1.3) in its operator equation form given by:

$$\begin{cases} N\left[\Pi(t;q)\right] = \Upsilon(t) \\ \text{with initial conditions } \Pi^{(k)}(0;1) = u^{(k)}(0) = \theta_k \text{ for } k = 0,1,\ldots,p-1, \text{ or} \\ \text{mixed boundary conditions } \Pi^{(k)}(0;1) = u^{(k)}(0) = \theta_k,\ k = 0,1,\ldots p-2; \\ \Pi(T;1) = u(T) = B. \end{cases} \tag{4.1}$$

With these conventions and letting $H(t) = 1$, the mth-order deformation equation of the standard HAM for (1.1) with initial conditions (1.2) becomes

$$\begin{cases} \hbar\left\{N[\Pi(t;q)] - \Upsilon(t)\right\}\Big|_{q=0}; \\ \pounds[u_m(t) - u_{m-1}(t)] = \hbar\Re_m(\overline{u}_{m-1}(t)) \text{ for } m = 2,3,\ldots; \\ u_0^{(k)}(0) = \theta_k \text{ for } k = 0,\ldots,p-1; \\ u_m^{(k)}(0) = 0 \text{ for } m = 1,2,\ldots \text{ and } k = 0,1,\ldots. \end{cases}$$

Alternatively, the standard HAM for (1.1) with boundary conditions (1.3) has the form

$$\begin{cases} \pounds[u_1(t)] = \hbar\Re_1(\overline{u}_0(t)) = \hbar\left\{N[\Pi(t;q)] - \Upsilon(t)\right\}\Big|_{q=0}; \\ \pounds[u_m(t) - u_{m-1}(t)] = \hbar\Re_m(\overline{u}_{m-1}(t)) \text{ for } m = 2,3,\ldots; \\ u_0^{(k)}(0) = \theta_k \text{ for } k = 0,\ldots,p-2; \\ u_m^{(k)}(0) = 0 \text{ for } m = 1,2,\ldots \text{ and } k = 0,1,\ldots; \\ u_0(T) = B; \quad u_m(T) = 0 \text{ for } m = 1,2,\ldots. \end{cases}$$

Here, the residual term $\Re_m(\overline{u}_{m-1}(t))$ is defined as in (3.5).
In a similar way, the ND-HAM equations for (1.1) with initial conditions (1.2) have the form

$$\begin{cases} \pounds[u_0(t)] = s_0(t); \\ \pounds[u_1(t)] = \hbar\Re_1(\overline{u}_0(t)) = \left\{N[\Pi(t;q)] - g(t;q)\right\}\Big|_{q=0}; \\ \pounds[u_m(t) - u_{m-1}(t)] = \hbar\Re_m(\overline{u}_{m-1}(t)) \text{ for } m = 2,3,\ldots; \\ u_0^{(k)}(0) = \theta_k \text{ for } k = 0,\ldots,p-1; \\ u_m^{(k)}(0) = 0 \text{ for } m = 1,2,\ldots \text{ and } k = 0,1,\ldots. \end{cases} \tag{4.2}$$

Alternatively, the ND-HAM for (1.1) with boundary conditions (1.3) is

$$\begin{cases} \pounds[u_0(t)] = s_0(t); \\ \pounds[u_1(t)] = \hbar\Re_1(\overline{u}_0(t)) = \hbar\left\{N[\Pi(t;q)] - g(t;q)\right\}\Big|_{q=0}; \\ \pounds[u_m(t) - u_{m-1}(t)] = \hbar\Re_m(\overline{u}_{m-1}(t)) \text{ for } m = 2,3,\ldots; \\ u_0^{(k)}(0)=\theta_k \text{ for } k=0,\ldots,p-2;\ \ u_m^{(k)}(0)=0 \text{ for } m=1,2,\ldots,\ k=0,1,\ldots; \\ u_0(T) = B; \quad u_m(T) = 0 \text{ for } m = 1,2,\ldots, \end{cases} \tag{4.3}$$

where, the residual term $\Re_m(\overline{u}_{m-1}(t))$ is defined as in (3.9).

We now implement the ND-HAM (4.3) for the boundary value problem (4.1). Since $\pounds = \frac{d^p}{dt^p}$ is linear differential operator of order p, by applying the inverse operator

$$\pounds^{-1}(\cdot) = J^{p}_{0+}(\cdot) = \frac{1}{(p-1)!}\int_0^t (t-\varsigma)^{p-1}(\cdot)d\varsigma,$$

on both sides of the equations of (4.3), and taking into account (2.2) and (2.3) with the conditions $u_m^{(k)}(0)=0$, for $m=1,2,\ldots$ and $k=0,1,\ldots$, we have

$$\begin{cases} u_0(t) = \sum_{k=0}^{p-2}\frac{\theta_k}{k!}t^k + \frac{C}{(p-1)!}t^{p-1} + \frac{1}{(p-1)!}\int_0^t (t-\varsigma)^{p-1}s_0(\varsigma)d\varsigma, \\ u_1(t) = \hbar J^p_{0+}\big[\big({}^c\mathfrak{D}^{\ell_p}_{0+} + \sum_{j=0}^{p-1}\vartheta_j\,{}^c\mathfrak{D}^{\ell_j}_{0+}\big)[u_0(t)] \\ \qquad -\big(\omega G_1(t,q)|_{q=0} + s_0(t) + s_1(t)\big)\big] \\ \quad = \hbar\big[J^{p-\ell_p}_{0+}\big(u_0(t) - \sum_{k=0}^{p-2}\frac{u_0^{(k)}(0)}{k!}t^k - \frac{C}{(p-1)!}t^{p-1}\big) \\ \qquad + \sum_{j=1}^{p-1}\vartheta_j{}^c J^{p-\ell_j}_{0+}\big(u_0(t) - \sum_{k=0}^{j-1}\frac{u_0^{(k)}(0)}{k!}t^k\big)\big] \\ \qquad + \frac{\hbar}{\Gamma(p)}\int_0^t (t-\varsigma)^{p-1}\Big[\omega G_1(\varsigma,q)|_{q=0} + s_0(\varsigma) + s_1(\varsigma)\Big]\,d\varsigma, \\ u_m(t) = u_{m-1}(t) + \hbar J^p_{0+}\Big[\big({}^c\mathfrak{D}^{\ell_p}_{0+} + \sum_{j=0}^{p-1}\vartheta_j{}^c\mathfrak{D}^{\ell_j}_{0+}\big)[u_{m-1}(t)] \\ \qquad -\big(\omega G_m(t,q)|_{q=0} + \hbar^{m-1}s_m(t)\big)\Big] \\ \quad = u_{m-1}(t) + \hbar\big(J^{p-\ell_p}_{0+}u_{m-1}(t) + \sum_{j=0}^{p-1}\vartheta_j{}^c J^{p-\ell_j}_{0+}u_{m-1}(t)\big) \\ \quad + \frac{\hbar}{\Gamma(p)}\int_0^t (t-\varsigma)^{p-1}\Big[\omega G_m(\varsigma,q)|_{q=0} + \hbar^{m-1}s_m(t)\Big]\,d\varsigma,\ 2\le m\le n, \\ u_m(t) = u_{m-1}(t) + \hbar\big(J^{p-\ell_p}_{0+}u_{m-1}(t) + \sum_{j=0}^{p-1}\vartheta_j{}^c J^{p-\ell_j}_{0+}u_{m-1}(t)\big) \\ \qquad + \frac{\hbar}{\Gamma(p)}\int_0^t (t-\varsigma)^{p-1}\Big[\omega G_m(\varsigma,q)|_{q=0}\Big]\,d\varsigma,\ \ m>n, \end{cases} \tag{4.4}$$

where $C = u_0^{(p-1)}(0)$ unknown parameter to be determined and

$$G_m(t,q) = \frac{1}{(m-1)!}\int_0^t\int_0^T \Lambda(x,s)\left[\frac{\partial^{m-1}\eth(\Pi(s;q))}{\partial q^{m-1}}\right]ds\,dx. \tag{4.5}$$

To find parameter C, we impose boundary conditions $u_0(T)=B$, which leads to

$$C = \frac{(p-1)!}{T}\Big[B - \sum_{k=0}^{p-2}\frac{\theta_k}{k!}t^k - \frac{1}{(p-1)!}\int_0^t (t-\varsigma)^{p-1}s_0(\varsigma)d\varsigma\Big].$$

Once we have known constant C, the next iteration of ND-HAM may be expressed as follows

$$\begin{cases} u_0(t) = \sum\limits_{k=0}^{p-2}\frac{\theta_k}{k!}t^k + \frac{C}{(p-1)!}t^{p-1} + \frac{1}{(p-1)!}\int_0^t (t-\varsigma)^{p-1}s_0(\varsigma)d\varsigma, \\ u_m(t) = u_{m-1}(t) - \frac{\hbar}{\Gamma(\eta-\ell_p)}\int_0^t (t-\varsigma)^{\eta-\ell_p-1}u_{m-1}(\varsigma)\,d\varsigma \end{cases}$$

$$\begin{cases} \quad +\hbar\sum_{j=1}^{p-1}\frac{\vartheta_j}{\Gamma(\eta-\ell_j)}\int_0^t (t-\varsigma)^{\eta-\ell_j-1}u_{m-1}(\varsigma)\,d\varsigma \\ \quad +\frac{\hbar}{\Gamma(p)}\int\limits_0^t (t-\varsigma)^{p-1}\Big[\omega G_m(\varsigma,q)|_{q=0}+\hbar^{m-1}s_m(t)\Big]\,d\varsigma,\ \ 1\le m\le n, \\ u_m(t)=u_{m-1}(t)-\frac{\hbar}{\Gamma(\eta-\ell_p)}\int_0^t (t-\varsigma)^{\eta-\ell_p-1}u_{m-1}(\varsigma)\,d\varsigma \\ \quad +\hbar\sum_{j=1}^{p-1}\frac{\vartheta_j}{\Gamma(\eta-\ell_j)}\int_0^t(t-\varsigma)^{\eta-\ell_j-1}u_{m-1}(\varsigma)\,d\varsigma \\ \quad -\frac{\omega\hbar}{\Gamma(\eta)}\int_0^t(t-\varsigma)^{\eta-1}G_m(\varsigma,q)|_{q=0}\,d\varsigma,\ \ m>n, \end{cases} \tag{4.6}$$

where G_m is defined by (4.5).

Special case. Provided that $\eth$ denotes a linear function provided that $\eth(u(t)) = u(t)$, the function G_m defined by (4.5) becomes

$$G_m(\varsigma,q)|_{q=0} = \int_0^{\varsigma}\int_0^{T}\Lambda(x,s)u_{m-1}(s)\,ds\,dx. \tag{4.7}$$

In this case, for $m\ge 2$, (4.6) yields

$$\begin{cases} u_m(t)=u_{m-1}(t)-\frac{\hbar}{\Gamma(\eta-\ell_p)}\int_0^t (t-\varsigma)^{\eta-\ell_p-1}u_{m-1}(\varsigma)\,d\varsigma \\ \quad +\hbar\sum_{j=1}^{p-1}\frac{\vartheta_j}{\Gamma(\eta-\ell_j)}\int_0^t(t-\varsigma)^{\eta-\ell_j-1}u_{m-1}(\varsigma)\,d\varsigma \\ \quad +\frac{\hbar}{\Gamma(p)}\int\limits_0^t (t-\varsigma)^{p-1}\Big[\omega G_m(\varsigma,q)|_{q=0}+\hbar^{m-1}s_m(t)\Big]\,d\varsigma,\ \ 1\le m\le n, \end{cases}$$

where $u_0(t)$ and $u_1(t)$ are defined in the first and second equation of (4.4) as well as nonlinear term $G_m(\varsigma,q)$ is computed in (4.7).

5 Uniqueness of the solution and convergence of the ND-HAM

This section proves the solutions' uniqueness with respect to the mixed nonlinear VF-IDEs of fractional order (1.1) with boundary conditions (1.3) and the convergence of the ND-HAM given by (4.3).

5.1 Uniqueness of the solution

For the sake of brevity, let us define the following constant:

$$\varrho = \frac{(p-1)!}{T^{p-1}}\Big\{B-\sum_{k=0}^{p-2}\frac{\theta_k T^k}{k!}-\sum_{j=1}^{p-1}\Big(\sum_{k=0}^{j-1}\frac{\vartheta_j\theta_k T^{\ell_p-\ell_j+p-1}}{\Gamma(\ell_p-\ell_j+k)}\Big)\Big\}. \tag{5.1}$$

Lemma 1. *Let $\Theta \in C([0,T],\mathbb{R})$ and $j-1<\ell_j\le j$ for $j=1,\dots,p$, where $p\ge 2$. Therefore, the FDE's solution is given by*

$$\Big({}^c\mathfrak{D}^{\ell_p}_{0^+} + \sum_{j=0}^{p-1} \vartheta_j \; {}^c\mathfrak{D}^{\ell_j}_{0^+}\Big) u(t) = \Theta(t), \tag{5.2}$$

with boundary conditions

$$u(T)=B \quad \text{and} \quad u^{(k)}(0)=\theta_k \text{ for } k=0,\dots,p-2,$$

is given by the integral equation

$$\begin{aligned} u(t) = {} & \frac{\varrho t^{p-1}}{(p-1)!} + \sum_{k=0}^{p-2} \frac{\theta_k t^k}{k!} + \sum_{j=1}^{p-1}\Big(\sum_{k=0}^{j-1} \frac{\vartheta_j \theta_k t^{\ell_p-\ell_j+p-1}}{\Gamma(\ell_p-\ell_j+k)}\Big) \\ & + \frac{t^{p-1}}{T^{p-1}} \sum_{j=1}^{p-1} \vartheta_j \; (J^{\ell_p-\ell_j}_{0^+} u)(T) - \sum_{j=1}^{p-1} \vartheta_j \; J^{\ell_p-\ell_j}_{0^+} u(t) \\ & + \frac{t^{p-1}}{T^{p-1}} (J^{\ell_p}_{0^+}\Theta)(T) - J^{\ell_p}_{0^+}\Theta(t). \end{aligned} \tag{5.3}$$

Proof. Upon implementing the operator $J^{\ell_p}_{0^+}$ with respect to both sides of (5.2), properties (2.1) and (2.3) yield

$$u(t) - \sum_{k=0}^{p-1} \frac{u^{(k)}(0)t^k}{k!} + \sum_{j=1}^{p-1} \vartheta_j \; J^{\ell_p-\ell_j}_{0^+}\Big(u(t) - \sum_{k=0}^{j-1}\frac{u^{(k)}(0)t^k}{k!}\Big) = J^{\ell_p}_{0^+}\Theta(t).$$

Following the boundary conditions $u^{(k)}(0)=\theta_k$ for $k=0,\dots,p-2$, and by (2.4), we obtain

$$\begin{aligned} u(t) - \frac{u^{(p-1)}(0)t^{p-1}}{(p-1)!} - \sum_{k=0}^{p-2}\frac{\theta_k t^k}{k!} + \sum_{j=1}^{p-1} \vartheta_j \; J^{\ell_p-\ell_j}_{0^+} u(t) \\ - \sum_{j=1}^{p-1}\Big(\sum_{k=0}^{j-1} \frac{\vartheta_j\theta_k t^{\ell_p-\ell_j+p-1}}{\Gamma(\ell_p-\ell_j+k)}\Big) = J^{\ell_p}_{0^+}\Theta(t). \end{aligned}$$

Thus,

$$\begin{aligned} u(t) = {} & \frac{c}{(p-1)!} t^{p-1} + \sum_{k=0}^{p-2} \frac{\theta_k}{k!} t^k + \sum_{j=1}^{p-1}\Big(\sum_{k=0}^{j-1}\frac{\vartheta_j\theta_k}{\Gamma(\ell_p-\ell_j+k)}\Big) t^{\ell_p-\ell_j+p-1} \\ & - \sum_{j=1}^{p-1} \vartheta_j \; J^{\ell_p-\ell_j}_{0^+} u(t) - J^{\ell_p}_{0^+}\Theta(t), \end{aligned} \tag{5.4}$$

where $c=u^{(p-1)}(0)$ can be determined using condition $u(T)=B$, which yields

$$c = \varrho + \frac{(p-1)!}{T^{p-1}} \sum_{j=1}^{p-1} \vartheta_j \; (J^{\ell_p-\ell_j}_{0^+} u)(T) + \frac{(p-1)!}{T^{p-1}} (J^{\ell_p}_{0^+}\Theta)(T),$$

with ϱ given by (5.1). Substituting this value into (5.4) yields (5.3). □

We now prove the solution's uniqueness to problem (1.1)–(1.3) employing the Banach contraction principle. Here, we can see that from Lemma 1, the VF-IDE provided is equivalent to the fractional integral equation given by

$$\begin{aligned} u(t) &= \frac{\varrho t^{p-1}}{(p-1)!} + \sum_{k=0}^{p-2} \frac{\theta_k t^k}{k!} + \sum_{j=1}^{p-1} \left(\sum_{k=0}^{j-1} \frac{\vartheta_j \theta_k t^{\ell_p - \ell_j + p - 1}}{\Gamma(\ell_p - \ell_j + k)} \right) \\ &+ \frac{t^{p-1}}{T^{p-1}} \sum_{j=1}^{p-1} \vartheta_j \left(J_{0^+}^{\ell_p - \ell_j} u \right)(T) - \sum_{j=1}^{p-1} \vartheta_j \, J_{0^+}^{\ell_p - \ell_j} u(t) \\ &+ \frac{t^{p-1}}{T^{p-1}} (J_{0^+}^{\ell_p} [\omega \widehat{G}(\cdot) + \Upsilon(\cdot)])(T) - J_{0^+}^{\ell_p} [\omega \widehat{G}(\cdot) + \Upsilon(\cdot)])(t), \end{aligned}$$

where

$$\widehat{G}(\cdot) = \int_0^{\cdot} \left(\int_0^T \Lambda(x,s) \eth(u(s)) \, ds \right) dx.$$

The operator $\Xi : X \to X$ may now be defined as

$$\begin{aligned} \Xi u(t) &= \frac{\varrho t^{p-1}}{(p-1)!} + \sum_{k=0}^{p-2} \frac{\theta_k t^k}{k!} + \sum_{j=1}^{p-1} \left(\sum_{k=0}^{j-1} \frac{\vartheta_j \theta_k t^{\ell_p - \ell_j + p - 1}}{\Gamma(\ell_p - \ell_j + k)} \right) \\ &+ \frac{t^{p-1}}{T^{p-1}} \sum_{j=1}^{p-1} \vartheta \int_0^T \frac{(T-\varsigma)^{\ell_p - \ell_j - 1}}{\Gamma(\ell_p - \ell_j)} u(\varsigma) \, d\varsigma - \sum_{j=1}^{p-1} \vartheta_j \int_0^t \frac{(t-\varsigma)^{\ell_p - \ell_j - 1}}{\Gamma(\ell_p - \ell_j)} u(\varsigma) \, d\varsigma \\ &+ \frac{t^{p-1}}{T^{p-1}} \int_0^T \frac{(T-\varsigma)^{\ell_p - 1}}{\Gamma(\ell_p)} \left[\omega \int_0^{\varsigma} \left(\int_0^T \Lambda(x,s) \eth(u(s)) \, ds \right) dx + \Upsilon(\varsigma) \right] d\varsigma \\ &- \int_0^t \frac{(t-\varsigma)^{\ell_p - 1}}{\Gamma(\ell_p)} \left[\omega \int_0^{\varsigma} \left(\int_0^T \Lambda(x,s) \eth(u(s)) \, ds \right) dx + \Upsilon(\varsigma) \right] d\varsigma. \end{aligned}$$

It is essential to highlight that u refers to a fixed point with respect to the operator Ξ if and only if u is a solution to problem (1.1)–(1.3).

Theorem 1. *Let $\eth : \mathbb{R} \to \mathbb{R}$ such that $\|\eth(x_1) - \eth(x_2)\| \le L \|x_1 - x_2\|$ for some $L > 0$ and all $(x_1, x_2) \in \mathbb{R}^2$. Then, problem (1.1)–(1.3) has a unique solution if*

$$\Delta = \sum_{j=1}^{p-1} \frac{|\vartheta_j| T^{\ell_p - \ell_j}}{\Gamma(\ell_p - \ell_j + 1)} + |\omega| \, L \, \|\Lambda\| \, \frac{T^{\ell_p + 2}}{\Gamma(\ell_p + 2)} < \frac{1}{2}, \tag{5.5}$$

with $\|\Lambda\| = \sup_{t,s \in [0,T]} \left| \Lambda(t,s) \right|.$

Proof. We now prove that Ξ is a contraction mapping. Hence, for every

$u, v \in X$ as well as $t \in [0, T]$, we now obtain

$$\begin{aligned}
\left|\varXi u(t) - \varXi v(t)\right| &= \Bigg| \frac{t^{p-1}}{T^{p-1}} \sum_{j=1}^{p-1} \vartheta_j \int_0^T \frac{(T-\varsigma)^{\ell_p-\ell_j-1}}{\Gamma(\ell_p-\ell_j)} \left[u(\varsigma) - v(\varsigma)\right] d\varsigma \\
&\quad - \sum_{j=1}^{p-1} \vartheta_j \int_0^t \frac{(t-\varsigma)^{\ell_p-\ell_j-1}}{\Gamma(\ell_p-\ell_j)} \left[u(\varsigma) - v(\varsigma)\right] d\varsigma \\
&\quad + \frac{t^{p-1}}{T^{p-1}} \int_0^T \frac{(T-\varsigma)^{\ell_p-1}}{\Gamma(\ell_p)} \left[\omega \int_0^\varsigma \left(\int_0^T \Lambda(x,s) \left[\eth(u(s)) - \eth(v(s))\right] ds \right) dx \right] d\varsigma \\
&\quad + \int_0^t \frac{(t-\varsigma)^{\ell_p-1}}{\Gamma(\ell_p)} \left[\omega \int_0^\varsigma \left(\int_0^T \Lambda(x,s) \left[\eth(u(s)) - \eth(v(s))\right] ds \right) dx \right] d\varsigma \Bigg| \\
&\le \|u-v\| \Bigg\{ \sum_{j=1}^{p-1} |\vartheta_j| \int_0^T \frac{(T-\varsigma)^{\ell_p-\ell_j-1}}{\Gamma(\ell_p-\ell_j)} d\varsigma + \sum_{j=1}^{p-1} |\vartheta_j| \int_0^t \frac{(t-\varsigma)^{\ell_p-\ell_j-1}}{\Gamma(\ell_p-\ell_j)} d\varsigma \\
&\quad + |\omega|\, L \,\|\Lambda\| \int_0^T \frac{(T-\varsigma)^{\ell_p-1}}{\Gamma(\ell_p)} \left[\int_0^\varsigma \left(\int_0^T ds \right) dx \right] d\varsigma \\
&\quad + |\omega|\, L \,\|\Lambda\| \int_0^t \frac{(t-\varsigma)^{\ell_p-1}}{\Gamma(\ell_p)} \left[\int_0^\varsigma \left(\int_0^T ds \right) dx \right] d\varsigma \Bigg\} \\
&= \Bigg\{ \sum_{j=1}^{p-1} \frac{|\vartheta_j| T^{\ell_p-\ell_j}}{\Gamma(\ell_p-\ell_j+1)} + \sum_{j=1}^{p-1} \frac{|\vartheta_j| t^{\ell_p-\ell_j}}{\Gamma(\ell_p-\ell_j+1)} + |\omega|\, L \,\|\Lambda\| \int_0^T \frac{(T-\varsigma)^{\ell_p-1}}{\Gamma(\ell_p)} [T\varsigma]\, d\varsigma \\
&\quad + |\omega|\, L \,\|\Lambda\| \int_0^t \frac{(t-\varsigma)^{\ell_p-1}}{\Gamma(\ell_p)} [T\varsigma]\, d\varsigma \Bigg\} \|u-v\|.
\end{aligned}$$

Upon implementing the change of variable $\varsigma = t - z$ with respect to the integral in the previous expression, we can see that

$$\begin{aligned}
\int_0^t \frac{T\varsigma(t-\varsigma)^{\ell_p-1}}{\Gamma(\ell_p)} d\varsigma &= \frac{T}{\Gamma(\ell_p)} \int_0^t \left(t z^{\ell_p-1} - z^{\ell_p} \right) dz = \frac{T t^{\ell_p+1}}{\Gamma(\ell_p)} \left[\frac{1}{\ell_p} - \frac{1}{\ell_p+1} \right] \\
&= \frac{T t^{\ell_p+1}}{\ell_p \left(\ell_p+1\right) \Gamma(\ell_p)} \le \frac{T^{\ell_p+2}}{\Gamma(\ell_p+2)}.
\end{aligned}$$

Then, for all $t \in [0, T]$, we have that

$$\left|\varXi u(t) - \varXi v(t)\right| \le \left\{ 2 \sum_{j=1}^{p-1} \frac{|\vartheta_j| T^{\ell_p-\ell_j}}{\Gamma(\ell_p-\ell_j+1)} + 2\,|\omega|\, L \,\|\Lambda\| \frac{T^{\ell_p+2}}{\Gamma(\ell_p+2)} \right\} \|u-v\|,$$

which, by definition of the norm, implies

$$\|\varXi u - \varXi v\| \le 2\Delta \|u-v\|.$$

With regards to the Banach contraction principle, the operator $\varXi$ possesses a unique fixed point. Thus, we may state a conclusion that problem (1.1)–(1.3) possesses a unique solution on the interval given by $[0, T]$. □

5.2 ND-HAM convergence

We now initially prove the ND-HAM convergence expressed by (4.3) with respect to the solution of (1.1) with boundary conditions (1.3).

Theorem 2 [Convergence theorem]. *Suppose the series $\sum_{m=0}^{\infty} u_m(t)$ converges to a function $u(t)$, in which the functions $u_m \in C(\mho, \mathbb{R})$ are governed by the high-order deformation equation (4.3) of the ND-HAM. Then, $u(t)$ defined by (4.3) approaches to the exact solution of the problem (1.1) with boundary conditions (1.3).*

Proof. Let us define $H_m(t) = \frac{1}{m!}\left[\frac{\partial^m \eth[\Pi(t;q)]}{\partial q^m}\right]_{q=0}$. It is illustrated in Cherruault [8] that, provided that the series $\sum_{m=0}^{\infty} u_m(t)$ approaches $u(t)$, then it is a must for the series $\sum_{k=0}^{\infty} H_k(t)$ to converge to $\eth[u(t)]$.
Let every $u_m(t)$ satisfies boundary conditions (1.3) i.e., $u_0^{(k)}(0) = \theta_k$, $k = 0, \ldots, p-2$ and $u_0(T) = B$ with $u_m^{(k)}(0) = 0$, $m = 1, 2, \ldots$, $k = 0, 1, \ldots$. Therefore, from the convergence with respect to $\sum_{m=0}^{\infty} u_m(t)$, it holds that

$$\lim_{m\to\infty} u_m(t) = 0,\ t \in [0, T]. \tag{5.6}$$

Summing on the left side with respect to (4.3) without the action operator $\pounds$ as well as considering (5.6), we now obtain

$$\sum_{m=1}^{\infty}\left[u_m(t) - \chi_m u_{m-1}(t)\right] = \lim_{n\to+\infty}\sum_{m=1}^{n}\left[u_m(t) - \chi_m u_{m-1}(t)\right] = \lim_{n\to+\infty} u_n(t) = 0.$$

Following the differential operator $\pounds = {}^c\mathfrak{D}_{0+}^{\eta}$, then (4.3) linearity implies

$$\begin{aligned}\hbar\sum_{m=1}^{+\infty}\Re_m(\overline{u}_{m-1}(t)) &= \sum_{m=1}^{+\infty} {}^c\mathfrak{D}_{0+}^{\eta}\left[u_m(t) - \chi_m u_{m-1}(t)\right]\\ &= \lim_{n\to\infty}\sum_{m=1}^{n} {}^c\mathfrak{D}_{0+}^{\eta}\left[u_m(t) - \chi_m u_{m-1}(t)\right]\\ &= \lim_{n\to\infty} {}^c\mathfrak{D}_{0+}^{\eta}[u_n(t)] = {}^c\mathfrak{D}_{0+}^{\eta}\left[\lim_{n\to\infty} u_n(t)\right] = {}^c\mathfrak{D}_{0+}^{\eta}(0) = 0.\end{aligned}$$

Since $\hbar \neq 0$, we must have

$$\sum_{m-1}^{+\infty}\Re_m(\overline{u}_{m-1}(t)) = 0.$$

Apart from that,

$$\begin{aligned}\Re_m(\overline{u}_{m-1}(t)) = &\left({}^c\mathfrak{D}_{0+}^{\ell_p} + \sum_{j=0}^{p-1}\vartheta_j\, {}^c\mathfrak{D}_{0+}^{\ell_j}\right)[u_{m-1}(t)] - (1-\chi_m)\Upsilon(t)\\ &- \frac{\omega}{(m-1)!}\int_0^t\int_0^T \Lambda(x,s)\left[\frac{\partial^{m-1}\eth[\Pi(s;q)]}{\partial q^{m-1}}\right]_{q=0} ds\,dx.\end{aligned}$$

Therefore, we have

$$0=\sum_{m=1}^{+\infty}\Re_m(\overline{u}_{m-1}(t))=\sum_{m=1}^{+\infty}\Bigg\{\Big({}^c\mathfrak{D}_{0^+}^{\ell_p}+\sum_{j=0}^{p-1}\vartheta_j\,{}^c\mathfrak{D}_{0^+}^{\ell_j}\Big)\Big[u_{m-1}(t)\Big]$$

$$-(1-\chi_m(t))\Upsilon(t)-\frac{\omega}{(m-1)!}\int_0^t\int_0^T\Lambda(x,s)\left[\frac{\partial^{m-1}\eth[\Pi(s;q)]}{\partial q^{m-1}}\right]_{q=0}ds\,dx\Bigg\}$$

$$=\sum_{m=1}^{+\infty}\Big({}^c\mathfrak{D}_{0^+}^{\ell_p}+\sum_{j=0}^{p-1}\vartheta_j\,{}^c\mathfrak{D}_{0^+}^{\ell_j}\Big)[u_{m-1}(t)]-\Upsilon(t)$$

$$-\omega\int_0^t\Big(\int_0^T\Lambda(s,x)\sum_{m=1}^{\infty}H_{m-1}(s)\,ds\Big)dx$$

$$=\Big({}^c\mathfrak{D}_{0^+}^{\ell_p}+\sum_{j=0}^{p-1}\vartheta_j\,{}^c\mathfrak{D}_{0^+}^{\ell_j}\Big)\Big[\sum_{m=1}^{\infty}u_{m-1}(t)\Big]-\Upsilon(t)$$

$$-\omega\int_0^t\Big(\int_0^T\Lambda(s,x)\sum_{m=1}^{\infty}H_{m-1}(s)\,ds\Big)dx$$

$$=\Big({}^c\mathfrak{D}_{0^+}^{\ell_p}+\sum_{j=0}^{p-1}\vartheta_j\,{}^c\mathfrak{D}_{0^+}^{\ell_j}\Big)u(t)-\Upsilon(t)-\omega\int_0^t\Big(\int_0^T\Lambda(s,x)\eth[u(s)]\,ds\Big)dx,$$

which implies

$$\Big({}^c\mathfrak{D}_{0^+}^{\ell_p}+\sum_{j=0}^{p-1}\vartheta_j\,{}^c\mathfrak{D}_{0^+}^{\ell_j}\Big)u(t)=\Upsilon(t)+\omega\int_0^t\int_0^T\Lambda(s,x)\eth[u(s)]\,ds\,dx.$$

It shows that if nonlinear FracIEs (1.1) with boundary conditions (1.3) has a unique solution, i.e., satisfies conditions of Theorem 3 (5.5), then ND-HAM defined by (4.3) convergence to exact solution. Theorem 4 is proved. □

6 Numerical experiments

Example 1. We now take into consideration the following nonlinear fractional IDEs given below:

$$\begin{cases}{}^c\mathfrak{D}_{0^+}^{\ell}u(t)=\Upsilon(t)+\omega\displaystyle\int_0^t\int_0^1(x-s)\eth(u(s))\,ds\,dx,\\ u(0)=1,u'(0)=0,1<\ell\le 2,t\in[0,1],\end{cases}\tag{6.1}$$

in which $\omega=1$, $\Upsilon(t)=-\dfrac{25}{504}t^2+\dfrac{749}{360}t$ as well as $\eth(u(t))=u^2(t)-u(t)$. It may be easily proven that $u(t)=\frac{1}{3}t^3+1$ is a solution of (6.1) for $\ell=2$.

Solution. Let us approximate the solution of Equation (6.1) using the ND-HAM given by (4.2) and (3.9). We can rewrite (6.1) in the operator form

$$\begin{cases}N(\Pi(t;q))=\Upsilon(t),\\ \Pi(0,1)=u(0)=1,\ \Pi'(0,1)=u'(0)=0,\end{cases}\tag{6.2}$$

where

$$N(\Pi(t;q)) = \ {}^{c}\mathfrak{D}_{0+}^{\ell}\Pi(t;q) - \int_0^t \int_0^1 (x-s)\eth(\Pi(s;q))\,ds\,dx,$$

for $1 < \ell \le 2$ and $t \in [0,1]$.

Uniqueness solution of the Example 1 can be checked easily. From (5.5) and given problem (6.1), we have

$$\Delta = |w|L\,\|\Lambda\|\,T^{\ell_2+2}/\Gamma(\ell_2+2),$$

where $\|\Lambda\| = \sup_{t,s\in[0,T]} \big|\Lambda(t,s)\big| = \sup_{t,s\in[0,T]} |x-s| = 1$, and $T = 1, \Gamma(\ell_2+2) = \Gamma(4) = 3! = 6$ and Lipschitz constant $L = \frac{5}{3}$, then $2\Delta = \frac{10}{18} < 1$, thus, satisfies the condition of Theorem 4.

Let us expand the right hand side function of (6.2) as

$$\Upsilon(t) = (2t) + \left(\frac{29}{360}t - \frac{25}{504}t^2\right) = s_0(t) + s_1(t) = [g(t;q)]_{q=0},$$

where $s_0(t) = 2t$ and $s_1(t) = \dfrac{29}{360}t - \dfrac{25}{504}t^2$. Since $\pounds = \dfrac{d^2}{dt^2}$ and solving the first equation of (4.2), we have

$$u_0(t) = 1 + t^3/3.$$

Let $m = 1$ and $\ell = 2$, then from the second equation with respect to (4.2), we may now possess

$$\begin{aligned}
\pounds[u_1(t)] &= \hbar[N(\Pi(t;q)) - g(t;q)]\,|_{q=0} \\
&= \hbar\Big[D_0^2[u_0(t)] - \int_0^t\int_0^1 (x-s)[u_0^2(s) - u_0(s)]\,ds\,dx - \Upsilon(t)\Big] \\
&= \hbar\left[2t - \left(\frac{25}{9\cdot 7\cdot 4}\frac{t^2}{2} - \frac{29}{9\cdot 8\cdot 5}t\right) - \left(\frac{749}{360}t - \frac{25}{504}t^2\right)\right] = \hbar[2t-2t] = 0,
\end{aligned}$$

which implies $u_1(t) = 0$. By continuing this procedure, we obtain $u_2(t) = u_3(t) = u_4(t) = \ldots = 0$. Therefore, the solution of (6.2) is

$$u(t) = \Pi(t;1) = \sum_{m=0}^{\infty} u_m(t) = u_0(t) = \frac{1}{3}t^3 + 1.$$

In Eshkuvatov et al. [9], it is shown that the modified HAM developed by Bataineh et al. [6] cannot coincide with the exact solution provided that the initial guess is chosen as the exact solution. As an alternative approach to the HAM, let us consider $1 < \ell \le 2$ and write the given function $\Upsilon(t)$ as

$$\Upsilon(t) = 0 + \frac{749}{360}t - \frac{25}{504}t^2 = s_0(t) + s_1(t) + s_2(t),$$

where $s_0(t)=0$, $s_1(t)=\frac{749}{360}t$, $s_2(t)=-\frac{25}{504}t^2$. Expanding $\Upsilon(t)$ into powers of the embedding parameter q yields

$$g(t;q)=s_0(t)+s_1(t)+(\hbar q)s_2(t).$$

From the first equation of (4.2), we have the initial guess $u_0(t)=1$. With this initial guess, the third iteration $n=3$ of the proposed ND-HAM [10], standard HAM [15] and modified HAM (MHAM) [6] at $\hbar=-1$, are given as follows:

- Exact solution: $u(t)=1+t^3/3$,
- ND-HAM: $u(t)\approx u_0(t)+u_1(t)+u_2(t)=-0.000521522266313933t^4 + 0.335200617283950616t^3+1$,
- HAM: $u(t)\approx u_0(t)+u_1(t)+u_2(t)=-0.0005559689153439t^4 + 0.33531543944738389t^3+1$,
- MHAM: $u(t)\approx u_0(t)+u_1(t)+u_2(t)=-0.000521522266313933t^4 + 0.335200617283950616t^3+1$.

Table 1 shows a comparison of the approximation errors for the different methods for $n=3$ iterations. As we can observe, the error of the ND-HAM is very close to the error of HAM.

Table 1. Numerical solution of Example 1 for HAM, MHAM, ND-HAM.

t	Exact	Err. HAM	Err. MHAM	Err. ND-HAM
	$n=3$	$n=3$	$n=3$	$n=3$
0.0	1.0000	0	0	0
0.2	1.0027	$1.4967\cdot10^{-5}$	$1.7460\cdot10^{-5}$	$1.7460\cdot10^{-5}$
0.4	1.0213	$1.1262\cdot10^{-4}$	$1.3121\cdot10^{-4}$	$1.3121\cdot10^{-4}$
0.6	1.0720	$3.5608\cdot10^{-4}$	$4.1428\cdot10^{-4}$	$4.1428\cdot10^{-4}$
0.8	1.1707	$7.8711\cdot10^{-4}$	$9.1428\cdot10^{-4}$	$9.1428\cdot10^{-4}$
1.0	1.3333	$1.4261\cdot10^{-3}$	$1.6534\cdot10^{-3}$	$1.6534\cdot10^{-3}$

Table 2. Numerical solution of Example 1 for HAM, MHAM, ND-HAM.

t	Exact	Err. HAM	Err. ND-HAM	Err. MHAM
	$n=15$	$n=15$	$n=15$	$n=15$
0.0	1.0000	0	0	0
0.2	1.0027	$1.9390\cdot10^{-16}$	$6.9948\cdot10^{-17}$	$6.9948\cdot10^{-17}$
0.4	1.0213	$1.4551\cdot10^{-15}$	$5.2660\cdot10^{-16}$	$5.2660\cdot10^{-16}$
0.6	1.0720	$4.5869\cdot10^{-15}$	$1.6659\cdot10^{-15}$	$1.6659\cdot10^{-15}$
0.8	1.1707	$1.0104\cdot10^{-11}$	$3.6850\cdot10^{-15}$	$3.6850\cdot10^{-15}$
1.0	1.3333	$1.8233\cdot10^{-14}$	$6.6819\cdot10^{-15}$	$6.6819\cdot10^{-15}$

Table 2 shows a comparison of the approximation errors for the different methods for $n=15$ iterations. Table 2 reveals that the error of the ND-HAM is slightly better than HAM for high iterations.

Remark 2. In Tables 1–2, we have compared methods (HAM, MHAM, ND-HAM) and chosen initial guess as $u_0(t) = 1$ with same number of iterations. Note that MHAM cannot give an exact solution provided that we select an initial guess as the solution with respect to problem (6.1). Providentially, HAM and ND-HAM provided an exact solution with the condition that if the initial guess is selected as $u_0 = 1 + \frac{t^3}{3}$, which is identical to the exact solution of problem (6.1). On the other hand, Table 2 shows that accuracy of ND-HAM is getting slightly better than HAM when number of iteration is increased.

Table 3 presents the comparison between HAM and ND-HAM for only three iterations for different value of fraction order of derivatives. From Figure 1, we can conclude that error of the method is decreasing when value of fractional derivative increases.

Table 3. Numerical solution of Example 1 for HAM and ND-HAM for different values of l_p.

t	Exact	Err. HAM	Err. ND-HAM	Err. HAM	Err. ND-HAM
	$n=3$	$n=3, l_p=1.5$	$n=3, l_p=1.5$	$n=3, l_p=1.95$	$n=3, l_p=1.95$
0.0	1.000000	0	0	0	0
0.2	1.002667	$-2.1438 \cdot 10^{-3}$	$-2.0689 \cdot 10^{-3}$	$-3.8516 \cdot 10^{-4}$	$-3.7424 \cdot 10^{-4}$
0.4	1.021333	$-1.4989 \cdot 10^{-2}$	$-1.4505 \cdot 10^{-2}$	$-2.3918 \cdot 10^{-3}$	$-2.3338 \cdot 10^{-3}$
0.6	1.072000	$-4.4986 \cdot 10^{-2}$	$-4.36459 \cdot 10^{-2}$	$-6.6621 \cdot 10^{-3}$	$-6.5256 \cdot 10^{-3}$
0.8	1.170667	$-9.5418 \cdot 10^{-2}$	$-9.2840 \cdot 10^{-2}$	$-1.3362 \cdot 10^{-2}$	$-1.3138 \cdot 10^{-2}$
1.0	1.333333	$-1.6708 \cdot 10^{-1}$	$-1.6304 \cdot 10^{-1}$	$-2.2349 \cdot 10^{-2}$	$-2.2055 \cdot 10^{-2}$

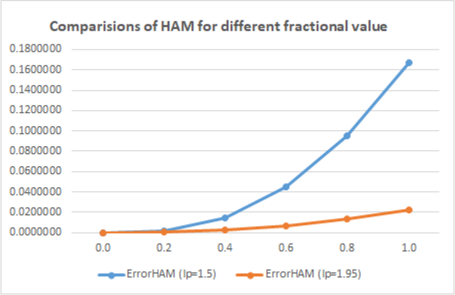

Figure 1. Comparisons the error term of HAM for lp= 1.5 and lp= 1.95.

Example 2. We may now take into consideration the following nonlinear fractional IDE given below:

$$\begin{cases} {}^c\mathfrak{D}^{\ell}_{0+} u(t) = \Upsilon(t) + \omega \displaystyle\int_0^t \int_0^1 (x-s)\eth(u(s))\,ds\,dx, \\ u(0) = 1, u'(0) = 0, 1 < \ell \le 2, t \in [0,1], \end{cases}$$

in which $\Upsilon(t) = 2 - \frac{21683}{1848}t - \frac{311}{840}t^2$ as well as $\eth(u(t)) = u^3(t)$ with $\omega = 1$. It can be easily verified that $u(t) = 1 + t^2 - 2t^3$ resembles a solution with respect to (6.2) for $\ell = 2$.

A summary of the results for Example 2 is tabulated in Table 4. Table 4 presents a comparison of the ND-HAM and standard HAM for $n = 3$ iterations and it reveals that the error for the ND-HAM is slightly better than standard HAM.

Table 4. Numerical solution of Example 2 for HAM, MHAM, ND-HAM.

t	Exact	Err. HAM	Err. MHAM	Err. ND-HAM
	$n = 3$	$n = 3$	$n = 3$	$n = 3$
0.0	1.0000	0	0	0
0.2	1.0240	$5.4448 \cdot 10^{-4}$	$5.1886 \cdot 10^{-4}$	$5.1886 \cdot 10^{-4}$
0.4	1.0320	$4.0952 \cdot 10^{-3}$	$3.9031 \cdot 10^{-3}$	$3.9031 \cdot 10^{-3}$
0.6	0.9280	$1.2941 \cdot 10^{-2}$	$1.2336 \cdot 10^{-2}$	$1.2336 \cdot 10^{-2}$
0.8	0.6160	$2.8590 \cdot 10^{-2}$	$2.7260 \cdot 10^{-2}$	$2.7260 \cdot 10^{-2}$
1.0	0.0000	$5.1767 \cdot 10^{-2}$	$4.9372 \cdot 10^{-2}$	$4.9372 \cdot 10^{-2}$

Table 5 presents a comparison of the ND-HAM and standard HAM for $n = 15$ iterations. As we can observe, the error for the ND-HAM is comparable with standard HAM and errors almost zero for both methods.

Table 5. Numerical solution of Example 2 for HAM, MHAM, ND-HAM.

t	Exact	Err. HAM	Err. MHAM	Err. ND-HAM
	$n = 15$	$n = 15$	$n = 15$	$n = 15$
0.0	1.0000	0	0	0
0.2	1.0240	$3.5644 \cdot 10^{-14}$	$3.6177 \cdot 10^{-14}$	$3.6177 \cdot 10^{-14}$
0.4	1.0320	$4.2456 \cdot 10^{-13}$	$5.2354 \cdot 10^{-13}$	$5.2354 \cdot 10^{-13}$
0.6	0.9280	$2.9985 \cdot 10^{-12}$	$3.2355 \cdot 10^{-12}$	$3.2355 \cdot 10^{-12}$
0.8	0.6160	$4.7657 \cdot 10^{-11}$	$4.9871 \cdot 10^{-11}$	$4.9871 \cdot 10^{-11}$
1.0	0.0000	$5.5492 \cdot 10^{-10}$	$5.7257 \cdot 10^{-10}$	$5.7257 \cdot 10^{-10}$

Example 3. Let us consider the Bagley-Torvik equation. This equation is a fractional ordinary differential equations resulting from the movement of a rigid plate immersed in a Newtonian fluid being modeled. In Roohollahi et al. [24] article, an integral term is added to the right equation in order to create a mixed Volterra-Fredholm integro-differential equation of fractional order:

$$\begin{cases} aD^l u(t) + b\ {}^c\mathfrak{D}_{0^+}^{3/2} u(t) + cu(t) = \Upsilon(t) + \omega \displaystyle\int_0^t \int_0^1 (1 - xs)u(s)\, ds\, dx, \\ u(0) = 0, u'(0) = -1,\ 1 < \ell \leq 2,\ t \in [0, 1], \end{cases}$$

where $\Upsilon(t) = \frac{2}{\sqrt{\pi}}t^{1/2} + \frac{11}{24}t^2 - \frac{1}{3}t + 2$ with $\omega = 1$. It can be easily verified that $u(t) = t^2 - t$ is the exact solution for $a = 1, b = c = 0.5$ and $\ell = 2$.

A summary of the results and comparisons with Roohollahi et al. [24] for Example 3 is tabulated in Table 6.

Table 6. Numerical solution of Example 2 for HAM, MHAM, ND-HAM.

t	Exact	Err. HAM	Err. NDHAM	Err. in [24]
	$n = 3$	$n = 3$	$n = 3$	$N = 64$
0.0	0.0000	0	0	0.00008138
0.2	−0.16	0.000078703443	0.000064875481	0.0018328
0.4	−0.24	0.000641911663	0.0005768473998	0.001228
0.6	−0.24	0.00226061057	0.001907843563	0.001897
0.8	−0.16	0.005671682234	0.004050550334	0.007541
1.0	0.000	0.01183676097	0.006377093824	0.015545

Here, $N = 64$ is the number of nodes on the $[0, 1]$ for the proposed method in [24]. Table 4 demonstrates a comparison of the ND-HAM, standard HAM for $n = 3$ iterations and proposed in Roohollahi et al. [24] with $N = 64$ named "Generalized block pulse operational differentiation matrices method". It reveals that the error for the ND-HAM is slightly better than standard HAM and "Generalized block pulse operational differentiation matrices method".

7 Conclusions

This work develops a novel HAM, called ND-HAM, to solve a class of nonlinear mixed VF-IDEs of fractional order. We proved the convergence of the HAM for this kind of equation and the uniqueness of the solution. In the example, we compared three methods (HAM, MHAM, and ND-HAM) for $n = 2$ and $n = 15$ iterations. From Tables 4 and 5, we can see that HAM, ND-HAM, and MHAM produce comparable convergence errors. Table 6 shows the domination of the proposed method over HAM and "Generalized block pulse operational differentiation matrices method" for multi-term of fractional Volterra-Fredholm IDEs. The numerical solutions were gained with the facilitation of the Matlab R2021b. These experiments show that the convergence is in line with theoretical results. The advantage of the ND-HAM is that, by choosing an appropriate u_0, the equations directly return the exact solution, as illustrated in both examples. Note that Examples 1–3 are nonlinear fractional IDEs.

Acknowledgements

Under the RMC Research Grant Scheme (UMT, 2020), Universiti Malaysia Terengganu (UMT) provided funding for this research. The project holds the project code 55233-UMT/CRIM/2-2/2/14 Jld. 4(44).

Abbreviations: The abbreviations given below are employed in this manuscript:

HAM	Homotopy Analysis Method
MHAM	Modified Homotopy Analysis Method
ND-HAM	New Development of the Hootopy Analysis Method
VF-IDEs	Volterra-Fredholm Integro-Differential Equations

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