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Joint Discrete Approximation of Analytic Functions by Shifts of Lerch Zeta-Functions

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Abstract. The Lerch zeta-function $L(\lambda,\alpha,s)$, $s=\sigma+it$, depends on two real parameters λ and $0<\alpha\leqslant 1$, and, for $\sigma>1$, is defined by the Dirichlet series $\sum_{m=0}^{\infty} e^{2\pi i\lambda m}(m+\alpha)^{-s}$, and by analytic continuation elsewhere. In the paper, we consider the joint approximation of collections of analytic functions by discrete shifts $(L(\lambda_1,\alpha_1,s+ikh_1),\dots,L(\lambda_r,\alpha_r,s+ikh_r))$, $k=0,1,\dots$, with arbitrary λ_j, $0<\alpha_j\leqslant 1$ and $h_j>0$, $j=1,\dots,r$. We prove that there exists a non-empty closed set of analytic functions on the critical strip $1/2<\sigma<1$ which is approximated by the above shifts. It is proved that the set of shifts approximating a given collection of analytic functions has a positive lower density. The case of positive density also is discussed. A generalization for some compositions is given.

Keywords: approximation of analytic functions, Lerch zeta-functions, space of analytic functions, weak convergence of probability measures.

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1 Introduction

The Lerch zeta-function $L(\lambda, \alpha, s)$, $s = \sigma + it$, with fixed parameters $\lambda \in \mathbb{R}$ and $0 < \alpha \leqslant 1$ is defined, in the half-plane $\sigma > 1$, by the Dirichlet series

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}}{(m+\alpha)^s}.$$

In virtue of periodicity of $e^{2\pi i \lambda m}$, it suffices to consider only the case $0 < \lambda \leqslant 1$. Clearly, $L(1, \alpha, s)$ coincides with the Hurwitz zeta-function

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s}, \quad \sigma > 1,$$

and $L(1, 1, s)$ is the Riemann zeta-function

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}, \quad \sigma > 1.$$

Therefore, in those cases, the function $L(\lambda, \alpha, s)$ has the analytic continuation to the whole complex plane, except for the point $s = 1$ which is a simple pole with residue 1. Moreover, the identities

$$L(1/2, 1, s) = \zeta(s)\left(1 - 2^{1-s}\right), \quad L(1, 1/2, s) = \zeta(s)\left(2^s - 1\right)$$

are valid. For $\lambda \notin \mathbb{Z}$, the function $L(\lambda, \alpha, s)$ is entire.

The function $L(\lambda, \alpha, s)$ was introduced in [22], and independently in [9]. Among other results for $L(\lambda, \alpha, s)$, M. Lerch proved in [22] the functional equation. Let $\Gamma(s)$ denote the Euler gamma-function. Then, for $0 < \lambda \leqslant 1$ and $s \in \mathbb{C}$,

$$\begin{aligned} L(\lambda, \alpha, 1-s) = &\frac{\Gamma(s)}{(2\pi)^s} \left(\exp\left\{ \frac{\pi i s}{2} - 2\pi i \alpha \lambda \right\} L(-\alpha, \lambda, s) \right. \\ &\left. + \exp\left\{ -\frac{\pi i s}{2} + 2\pi i \alpha (1-\lambda) \right\} L(\alpha, 1-\lambda, s) \right). \end{aligned}$$

Another proofs of the functional equation for $L(\lambda, \alpha, s)$ were proposed by B.C. Berndt [5] and T.M. Apostol [1, 2]. The above and other analytic results on the function $L(\lambda, \alpha, s)$ also can be found in [15]. In general, the Lerch zeta-function is an important object of analytic number theory, and appears in solving many problems of mathematics. In particular, the function $L(\lambda, \alpha, s)$ is useful in the theory of special functions. On the other hand, the Lerch zeta-function is an interesting analytic object and is studied by analytic number theorists. Approximation problems of analytic functions by shifts of $L(\lambda, \alpha, s + i\tau)$, $\tau \in \mathbb{R}$, is one of directions of investigations of the function $L(\lambda, \alpha, s)$. We recall that the idea of approximation of analytic functions by shifts of zeta-functions belongs to S.M. Voronin who opened this problem in [33] for the Riemann zeta-function and Dirichlet L-functions, and called it universality, see also [10].

Voronin's ideas were developed by numerous authors, see [3, 8, 12, 23, 32]. The first result on universality of the Lerch zeta-function was obtained in [13], see also [15]. Let $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$. Denote by $\mathcal{K}$ the class of compact subsets of the strip D with connected complements, and by $H(K)$ with $K \in \mathcal{K}$ the class of continuous functions on K that are analytic in the interior of K. Let meas$A$ stand for the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Then the theorem of [13] is the following statement.

Theorem 1. *Suppose that α is a transcendental number, and $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas}\Big\{\tau \in [0,T] : \sup_{s\in K} |L(\lambda,\alpha,s+i\tau) - f(s)| < \varepsilon\Big\} > 0.$$

We notice that the form of Theorem 1 extends that of the Voronin theorem in two directions. First, he approximated analytic functions only on discs of the strip D by shifts $\zeta(s+i\tau)$. Secondly, Voronin claimed that there exists $\tau \in \mathbb{R}$ such that $\zeta(s+i\tau)$ approximates a given function $f(s)$, while, by Theorem 1, there exist infinitely many shifts $L(\lambda,\alpha,s+i\tau)$ approximating $f(s)$. A weighted version of Theorem 1 was obtained in [7].

Theorem 1 has its discrete version. In this case, τ runs over a certain discrete set. Such a version of universality was proposed by A. Reich in [30] for Dedekind zeta-functions. A discrete universality theorem for the function $L(\lambda,\alpha,s)$ follows from a more general similar theorem for the periodic Hurwitz zeta-function obtained in [16]. Denote by $\#A$ the number of elements of the set $A \subset \mathbb{R}$. Then we have

Theorem 2. *[16] Suppose that the parameter λ is rational, the parameter α is a transcendental, and the number $h > 0$ is such that the number* $\exp\{(2\pi)/h\}$ *is rational. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\Big\{0 \leqslant k \leqslant N : \sup_{s\in K} |L(\lambda,\alpha,s+ikh) - f(s)| < \varepsilon\Big\} > 0.$$

Observe that Theorem 2 has a certain advantage against Theorem 1 because a detection of approximating shifts in discrete set is easier than in a full interval in the case of Theorem 1.

Theorems 1 and 2 have joint generalizations on simultaneous approximation of a collection of analytic functions. In this case, the important role is played by a certain independence of shifts $L(\lambda_j,\alpha_j,s+i\tau)$ or $L(\lambda_j,\alpha_j,s+ikh)$. For example, in [17, 18, 19, 21, 25, 27, 28, 29], the algebraic independence of the parameters $\alpha_1,\dots,\alpha_r$ was applied. Recall a joint discrete universality theorem for Lerch zeta-functions. For $h > 0$, define the set

$$L(\alpha_1,\dots,\alpha_r;h,\pi) = \Big\{\big(\log(m+\alpha_1) : m \in \mathbb{N}_0\big),\dots,\big(\log(m+\alpha_r) : m \in \mathbb{N}_0\big), 2\pi/h\Big\},$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Then, in [19], the following assertion was proved.

Theorem 3. *Suppose that the set $L(\alpha_1,\dots,\alpha_r;h,\pi)$ is linearly independent over the field of rational numbers $\mathbb{Q}$. For $j=1,\dots,r$, let $K_j\in\mathcal{K}$ and $f_j(s)\in H(K_j)$, and $0<\lambda_j\leqslant 1$. Then, for every $\varepsilon>0$,*

$$\liminf_{N\to\infty}\frac{1}{N+1}\#\left\{0\leqslant k\leqslant N:\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|L(\lambda_j,\alpha_j,s+ikh)-f_j(s)|<\varepsilon\right\}>0.$$

Moreover, "$\liminf$*" can be replaced by "*$\lim$*" for all but at most countably many $\varepsilon>0$.*

All stated or mentioned above theorems are valid for some classes of parameters λ and $0<\alpha\leqslant 1$. A question arises do the above results remain valid for all values of parameters λ and $0<\alpha\leqslant 1$. Unfortunately, this question is an open problem. In [14, 20, 31], a certain type of approximation of analytic functions by shifts of Lerch zeta-function with all parameters λ and α was proposed. This type is not universality but shows good approximation properties of the function $L(\lambda,\alpha,s)$. We recall a discrete version of approximation from [31]. Denote by $H(D)$ the space of analytic on D functions endowed with the topology of uniform convergence on compacta.

Theorem 4. *[31] Suppose that the parameters λ, α and the number $h>0$ are arbitrary. Let K be a compact set of the strip D. Then there exists a closed non-empty set $F_{\lambda,\alpha,h}\subset H(D)$ such that, for $f(s)\in F_{\lambda,\alpha,h}$ and $\varepsilon>0$,*

$$\liminf_{N\to\infty}\frac{1}{N+1}\#\left\{0\leqslant k\leqslant N:\sup_{s\in K}|L(\lambda,\alpha,s+ikh)-f(s)|<\varepsilon\right\}>0.$$

Moreover, "$\liminf$*" can be replaced by "*$\lim$*" for all but at most countably many $\varepsilon>0$.*

Here and in the sequel, "arbitrary α" means that α satisfies $0<\alpha\leqslant 1$.

The aim of this paper is a joint version of Theorem 4. Denote

$$H^r(D)=\underbrace{H(D)\times\cdots\times H(D)}_{r}.$$

The space $H^r(D)$ is metrisable. Let $\{K_l:l\in\mathbb{N}\}\subset D$ be a sequence of compact embedded sets such that $D=\bigcup_{l=1}^{\infty}K_l$, and, for every compact set $K\subset D$, there exists K_l such that $K\subset K_l$. Then, putting

$$\rho(g_1,g_2)=\sum_{l=1}^{\infty}2^{-l}\frac{\sup_{s\in K_l}|g_1(s)-g_2(s)|}{1+\sup_{s\in K_l}|g_1(s)-g_2(s)|},\quad g_1,g_2\in H(D),$$

we have a metric which induces the topology of uniform convergence on compacta of the space $H(D)$. Then,

$$\underline{\rho}(\underline{g}_1,\underline{g}_2)=\max_{1\leqslant j\leqslant r}\rho(g_{1j},g_{2j}),\quad \underline{g}_k=(g_{k1},\dots,g_{kr}),\ k=1,2,$$

is a metric inducing the product topology of $H^r(D)$.

The main result of the paper is the following theorem. Let $\underline{\lambda}=(\lambda_1,\dots,\lambda_r)$, $\underline{\alpha}=(\alpha_1,\dots,\alpha_r)$ and $\underline{h}=(h_1,\dots,h_r)$.

Theorem 5. *Suppose that the parameters* λ_j *and* α_j*, and* $h_j > 0$, $j = 1, \dots, r$, *are arbitrary. Then there exists a non-empty closed set* $F_{\underline{\lambda},\underline{\alpha},\underline{h}} \subset H^r(D)$ *such that, for compact sets* $K_1, \dots, K_r$ *of* D, $(f_1(s), \dots, f_r(s)) \in F_{\underline{\lambda},\underline{\alpha},\underline{h}}$ *and* $\varepsilon > 0$,

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\Big\{0 \leqslant k \leqslant N : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |L(\lambda_j, \alpha_j, s + ikh_j) - f_j(s)| < \varepsilon\Big\} > 0.$$

Moreover, "$\liminf$*" can be replaced by "*$\lim$*" for all but at most countably many* $\varepsilon > 0$.

Let $\underline{L}(\underline{\lambda}, \underline{\alpha}, s) = (L(\lambda_1, \alpha_1, s), \dots, L(\lambda_r, \alpha_r, s))$. Theorem 5 can be generalized for certain compositions $\Psi(\underline{L}(\underline{\lambda}, \underline{\alpha}, s))$. We give one example.

Theorem 6. *Suppose that the parameters* λ_j *and* α_j*, and* $h_j > 0, j = 1, \dots, r$, *are arbitrary. Then there exists a non-empty closed set* $F_{\underline{\lambda},\underline{\alpha},\underline{h}} \subset H^r(D)$ *such that if* $\Psi : H^r(D) \to H(D)$ *is a continuous operator such that, for every polynomial* $p = p(s)$*, the set* $(\Psi^{-1}\{p\}) \cap F_{\underline{\lambda},\underline{\alpha},\underline{h}}$ *is non-empty, then, for every compact set* $K \subset D$, $f(s) \in \Psi(F_{\underline{\lambda},\underline{\alpha},\underline{h}})$ *and* $\varepsilon > 0$,

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\Big\{0 \leqslant k \leqslant N : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\Psi(\underline{L}(\underline{\lambda}, \underline{\alpha}, s + ik\underline{h}) - f(s)| < \varepsilon\Big\} > 0.$$

Moreover, "$\liminf$*" can be replaced by "*$\lim$*" for all but at most countably many* $\varepsilon > 0$.

To prove Theorems 5 and 6, we will obtain a probabilistic limit theorem for $\underline{L}(\underline{\lambda}, \underline{\alpha}, s)$ in the space $H^r(D)$. The support of the limit measure in that theorem will be desired set $F_{\underline{\lambda},\underline{\alpha},\underline{h}}$. Theorem 5 covers the results of [4] obtained for Hurwitz zeta-functions. Joint discrete approximation by shifts of more general zeta-functions is given in [11].

2 A limit theorem on a group

Denote by $\mathcal{B}(\mathbb{X})$ the Borel σ-field of a topological space $\mathbb{X}$. Our final aim is a limit theorem for

$$P_{N,\underline{\lambda},\underline{\alpha},\underline{h}}(A) = \frac{1}{N+1} \# \left\{0 \leqslant k \leqslant N : \underline{L}(\underline{\lambda}, \underline{\alpha}, s + ik\underline{h}) \in A\right\}, \quad A \in \mathcal{B}(H^r(D)),$$

as $N \to \infty$. We divide a proof of this theorem into lemmas, and the first of them is a limit lemma on the r-dimensional torus. Define $\Omega = \prod_{m\in\mathbb{N}_0} \gamma_m$, where $\gamma_m = \{s \in \mathbb{C} : |s| = 1\}$ for all $m \in \mathbb{N}_0$. With the product topology and operation of pointwise multiplication, the torus Ω is a compact topological Abelian group. Set $\Omega^r = \prod_{j=1}^r \Omega_j$, where $\Omega_j = \Omega$ for all $j = 1, \dots, r$. Then, by the Tikhonov theorem, Ω^r again is a compact topological Abelian group. For $A \in \mathcal{B}(\Omega^r)$, define

$$\begin{aligned} Q_{N,\underline{\alpha},\underline{h}}(A) = \frac{1}{N+1} \#\Big\{0 \leqslant k \leqslant N :&(((m + \alpha_1)^{-ikh_1} : m \in \mathbb{N}_0), \dots, \\ &((m + \alpha_r)^{-ikh_r} : m \in \mathbb{N}_0)) \in A\Big\}. \end{aligned}$$

Lemma 1. *Suppose that $\underline{\alpha}$ and $\underline{h}$ are arbitrary. Then, on $(\Omega^r, \mathcal{B}(\Omega^r))$, there exists a probability measure $Q_{\underline{\alpha},\underline{h}}$ such that $Q_{N,\underline{\alpha},\underline{h}}$ converges weakly to $Q_{\underline{\alpha},\underline{h}}$ as $N \to \infty$.*

Proof. Proofs of limit theorems on compact groups usually are based on continuity theorems for Fourier transformations. Denote by $\omega_j(m)$ the mth component of an element of $\omega_j \in \Omega_j$, $j = 1, \dots, r$, $m \in \mathbb{N}_0$. Then the characters of Ω^r are of the form

$$\chi(\omega) = \prod_{j=1}^{r} \prod_{m\in\mathbb{N}_0}{}^{*} \omega_j^{k_{jm}}(m),$$

where $\omega = (\omega_1, \dots, \omega_r)$ denotes an element of Ω^r, and the sign "$*$" indicate that only a finite number of integers k_{jm} are distinct from zero. Hence, the Fourier transform $g_{N,\underline{\alpha},\underline{h}}(\underline{k}_1, \dots, \underline{k}_r)$, $\underline{k}_j = (k_{jm} : k_{jm} \in \mathbb{Z},\, m \in \mathbb{N}_0)$, $j = 1, \dots, r$, has the representation

$$g_{N,\underline{\alpha},\underline{h}}(\underline{k}_1, \dots, \underline{k}_r) = \int_{\Omega^r} \Big(\prod_{j=1}^{r} \prod_{m\in\mathbb{N}_0}{}^{*} \omega_j^{k_{jm}}(m) \Big) \, \mathrm{d}Q_{N,\underline{\alpha},\underline{h}}.$$

Thus, the definition of $Q_{N,\underline{\alpha},\underline{h}}$ gives

$$\begin{aligned} g_{N,\underline{\alpha},\underline{h}}(\underline{k}_1, \dots, \underline{k}_r) &= \frac{1}{N+1} \sum_{k=0}^{N} \prod_{j=1}^{r} \prod_{m\in\mathbb{N}_0}{}^{*} (m+\alpha_j)^{-ikh_j k_{jm}} \\ &= \frac{1}{N+1} \sum_{k=0}^{N} \exp\Big\{ -ik \sum_{j=1}^{r} h_j \sum_{m\in\mathbb{N}_0}{}^{*} k_{jm} \log(m+\alpha_j) \Big\}. \end{aligned} \tag{2.1}$$

Define two sets of tuples $(\underline{k}_1, \dots, \underline{k}_r)$. Let

$$A_{1,\underline{\alpha},\underline{h}} = \Big\{ (\underline{k}_1, \dots, \underline{k}_r) : \sum_{j=1}^{r} h_j \sum_{m\in\mathbb{N}_0}{}^{*} k_{jm} \log(m+\alpha_j) = 2\pi l,\ \exists l \in \mathbb{Z} \Big\}$$

$$A_{2,\underline{\alpha},\underline{h}} = \Big\{ (\underline{k}_1, \dots, \underline{k}_r) : \sum_{j=1}^{r} h_j \sum_{m\in\mathbb{N}_0}{}^{*} k_{jm} \log(m+\alpha_j) \neq 2\pi l \text{ for every } l \in \mathbb{Z} \Big\}.$$

Then, clearly, for $(\underline{k}_1, \dots, \underline{k}_r) \in A_{1,\underline{\alpha},\underline{h}}$, equality (2.1) implies

$$g_{N,\underline{\alpha},\underline{h}}(\underline{k}_1, \dots, \underline{k}_r) = 1,$$

while, for $(\underline{k}_1, \dots, \underline{k}_r) \in A_{2,\underline{\alpha},\underline{h}}$, we have

$$g_{N,\underline{\alpha},\underline{h}}(\underline{k}_1, \dots, \underline{k}_r) = \frac{1 - \exp\Big\{ -(N+1)i \sum_{j=1}^{r} h_j \sum^{*}_{m\in\mathbb{N}_0} k_{jm} \log(m+\alpha_j) \Big\}}{(N+1)\Big(1 - \exp\Big\{ -i \sum_{j=1}^{r} h_j \sum^{*}_{m\in\mathbb{N}_0} k_{jm} \log(m+\alpha_j) \Big\}\Big)}.$$

This together with (2.1) shows that

$$\lim_{N\to\infty} g_{N,\underline{\alpha},\underline{h}}(\underline{k}_1, \dots, \underline{k}_r) = g_{\underline{\alpha},\underline{h}}(\underline{k}_1, \dots, \underline{k}_r), \tag{2.2}$$

where

$$g_{\underline{\alpha},\underline{h}}(\underline{k}_1,\dots,\underline{k}_r)=\begin{cases}1 & \text{if } (\underline{k}_1,\dots,\underline{k}_r)\in A_{1,\underline{\alpha},\underline{h}},\\ 0 & \text{if } (\underline{k}_1,\dots,\underline{k}_r)\in A_{2,\underline{\alpha},\underline{h}}.\end{cases}$$

Denote by $Q_{\underline{\lambda},\underline{\alpha}}$ the probability measure on $(\Omega^r,\mathcal{B}(\Omega^r))$ defined by the Fourier transform $g_{\underline{\alpha},\underline{h}}(\underline{k}_1,\dots,\underline{k}_r)$. Then, in view of (2.2), we obtain that $Q_{N,\underline{\lambda},\underline{\alpha}}$ converges weakly to the measure $Q_{\underline{\lambda},\underline{\alpha}}$ as $N\to\infty$. The lemma is proved. □

For example, if the set

$$\{(h_1\log(m+\alpha_1):m\in\mathbb{N}_0),\dots,(h_r\log(m+\alpha_r):m\in\mathbb{N}_0),2\pi\}$$

is linearly independent over $\mathbb{Q}$, then,

$$g_{N,\underline{\alpha},\underline{h}}(\underline{k}_1,\dots,\underline{k}_r)=\begin{cases}1 & \text{if } (\underline{k}_1,\dots,\underline{k}_r)=(\underline{0},\dots,\underline{0}),\\ 0 & \text{if } (\underline{k}_1,\dots,\underline{k}_r)\neq(\underline{0},\dots,\underline{0}).\end{cases}$$

Therefore, in this case, $Q_{N,\underline{\alpha},\underline{h}}$ converges weakly to the probability Haar measure m_H on $(\Omega^r,\mathcal{B}(\Omega^r))$ as $N\to\infty$.

Lemma 1 allows to consider weak convergence for probability measures defined by means of absolutely convergent Dirichlet series. Define

$$P_{N,n,\underline{\lambda},\underline{\alpha},\underline{h}}(A)=\frac{1}{N+1}\#\{0\leqslant k\leqslant N:\underline{L}_n(\underline{\lambda},\underline{\alpha},s+ik\underline{h})\in A\},A\in\mathcal{B}(H^r(D)),$$

where

$$\underline{L}_n(\underline{\lambda},\underline{\alpha},s)=(L_n(\lambda_1,\alpha_1,s),\dots,L_n(\lambda_r,\alpha_r,s))$$

with

$$L_n(\lambda_j,\alpha_j,s)=\sum_{m=0}^{\infty}\frac{\mathrm{e}^{2\pi i\lambda_j m}v_n(m,\alpha_j)}{(m+\alpha_j)^s},\quad j=1,\dots,r,$$
$$v_n(m,\alpha_j)=\exp\left\{-\left((m+\alpha_j)/n\right)^{\theta}\right\},\quad \theta>1/2.$$

Obviously, the series for $L_n(\lambda_j,\alpha_j,s)$ are absolutely convergent, say, for $\sigma>0$.

Lemma 2. *Suppose that $\underline{\lambda}$, $\underline{\alpha}$ and $\underline{h}$ are arbitrary. Then, on $(H^r(D),\mathcal{B}(H^r(D)))$, there exists a probability measure $\widehat{P}_{n,\underline{\lambda},\underline{\alpha},\underline{h}}$ such that $P_{N,n,\underline{\lambda},\underline{\alpha},\underline{h}}$ converges weakly to $\widehat{P}_{n,\underline{\lambda},\underline{\alpha},\underline{h}}$ as $N\to\infty$.*

Proof. For $\omega\in\Omega^r$, define

$$\underline{L}_n(\underline{\lambda},\underline{\alpha},\omega,s)=(L_n(\lambda_1,\alpha_1,\omega_1,s),\dots,L_n(\lambda_r,\alpha_r,\omega_r,s)),$$

where

$$L_n(\lambda_j,\alpha_j,\omega_j,s)=\sum_{m=0}^{\infty}\frac{\mathrm{e}^{2\pi i\lambda_j m}\omega_j(m)v_n(m,\alpha_j)}{(m+\alpha_j)^s},\quad j=1,\dots,r.$$

Let the mapping $u_{n,\underline{\lambda},\underline{\alpha}} : \Omega^r \to H^r(D)$ be given by the formula

$$u_{n,\underline{\lambda},\underline{\alpha}}(\omega) = \underline{L}_n(\underline{\lambda}, \underline{\alpha}, \omega, s).$$

Since the series defining $\underline{L}_n(\underline{\lambda}, \underline{\alpha}, \omega, s)$, as $\underline{L}_n(\underline{\lambda}, \underline{\alpha}, s)$, are absolutely convergent in the strip D, the mapping $u_{n,\underline{\lambda},\underline{\alpha}}$ is continuous, hence, it is $(\mathcal{B}(\Omega^r), \mathcal{B}(H^r(D)))$-measurable. Moreover, by the definitions of $P_{N,n,\underline{\lambda},\underline{\alpha},\underline{h}}$, $Q_{N,\underline{\alpha},\underline{h}}$ and $u_{n,\underline{\lambda},\underline{\alpha}}$, we have

$$u_{n,\underline{\lambda},\underline{\alpha}} \left(\left((m+\alpha_1)^{-ikh_1} : m \in \mathbb{N}_0 \right), \ldots, \left((m+\alpha_r)^{-ikh_r} : m \in \mathbb{N}_0 \right) \right) \\ = \underline{L}_n(\underline{\lambda}, \underline{\alpha}, \omega, s)$$

and, for $A \in \mathcal{B}(H^r(D))$,

$$P_{N,n,\underline{\lambda},\underline{\alpha},\underline{h}}(A) = \frac{1}{N+1} \# \Big\{ 0 \leqslant k \leqslant N : \big(\left((m+\alpha_1)^{-ikh_1} : m \in \mathbb{N}_0 \right), \ldots, \\ \left((m+\alpha_r)^{-ikh_r} : m \in \mathbb{N}_0 \right) \big) \in u^{-1}_{n,\underline{\lambda},\underline{\alpha}} A \Big\} = Q_{N,\underline{\alpha},\underline{h}} \left(u^{-1}_{n,\underline{\lambda},\underline{\alpha}} A \right) = Q_{N,\underline{\alpha},\underline{h}} u^{-1}_{n,\underline{\lambda},\underline{\alpha}}(A).$$

Therefore, $P_{N,n,\underline{\lambda},\underline{\alpha},\underline{h}} = Q_{N,\underline{\alpha},\underline{h}} u^{-1}_{n,\underline{\lambda},\underline{\alpha}}$. Since the mapping $u_{n,\underline{\lambda},\underline{\alpha}}$ is $(\mathcal{B}(\Omega^r), \mathcal{B}(H^r(D)))$-measurable, the measures $Q_{N,\underline{\alpha},\underline{h}} u^{-1}_{n,\underline{\lambda},\underline{\alpha}}$ and $Q_{\underline{\alpha},\underline{h}} u^{-1}_{n,\underline{\lambda},\underline{\alpha}}$ are well defined. These remarks, Lemma 1 and a property of preservation of weak convergence under continuous mappings, see, for example, Theorem 5.1 of [6], show that $P_{N,n,\underline{\lambda},\underline{\alpha},\underline{h}}$ converges weakly to the probability measure $\widehat{P}_{n,\underline{\lambda},\underline{\alpha},\underline{h}} \stackrel{\text{def}}{=} Q_{\underline{\alpha},\underline{h}} u^{-1}_{n,\underline{\lambda},\underline{\alpha}}$ as $N \to \infty$. □

3 Distance between $\underline{L}(\underline{\lambda}, \underline{\alpha}, s)$ and $\underline{L}_n(\underline{\lambda}, \underline{\alpha}, s)$

In view of Lemma 2, to prove a limit theorem for $P_{N,\underline{\lambda},\underline{\alpha},\underline{h}}$ it is sufficient to show that the distance between $\underline{L}(\underline{\lambda}, \underline{\alpha}, s)$ and $\underline{L}_n(\underline{\lambda}, \underline{\alpha}, s)$ in the space $H^r(D)$ is small. For this, we apply the following lemma obtained in [31].

Lemma 3. *The equality*

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \rho \left(L(\lambda, \alpha, s+ikh), L_n(\lambda, \alpha, s+ikh) \right) = 0$$

holds for all λ, α *and* $h > 0$.

We recall that, for the proof of Lemma 3, the mean square estimates

$$\int_{-T}^{T} |L(\lambda, \alpha, \sigma+it)|^2 \, \mathrm{d}t \ll_{\lambda,\alpha,\sigma} T, \quad T > 0, \tag{3.1}$$

$$\int_{-T}^{T} |L'(\lambda, \alpha, \sigma+it)|^2 \, \mathrm{d}t \ll_{\lambda,\alpha,\sigma} T, \quad T > 0, \tag{3.2}$$

for $1/2 < \sigma < 1$, the Gallagher lemma, see Lemma 1.4 of [26], connecting the discrete and continuous mean squares, and the integral representation [15]

$$L_n(\lambda, \alpha, s) = \frac{1}{2\pi i} \int_{\theta-i\infty}^{\theta+i\infty} \frac{1}{\theta} L(\lambda, \alpha, s+z) \Gamma\left(\frac{z}{\theta}\right) n^z \, \mathrm{d}z$$

are applied.

Lemma 4. *The equality*

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\underline{\rho}\left(\underline{L}(\underline{\lambda},\underline{\alpha},s+ik\underline{h}),\underline{L}_n(\underline{\lambda},\underline{\alpha},s+ik\underline{h})\right)=0$$

holds for all $\underline{\lambda}$, $\underline{\alpha}$ *and* $\underline{h}>0$.

Proof. By the definition of the metric $\underline{\rho}$,

$$\sum_{k=0}^{N}\underline{\rho}\left(\underline{L}(\underline{\lambda},\underline{\alpha},s+ik\underline{h}),\underline{L}_n(\underline{\lambda},\underline{\alpha},s+ik\underline{h})\right)$$
$$\leqslant\sum_{j=1}^{r}\sum_{k=0}^{N}\rho\left(L(\lambda_j,\alpha_j,s+ikh_j),L_n(\lambda_j,\alpha_j,s+ikh_j)\right).$$

Therefore, the lemma is a corollary of Lemma 3. □

4 Relative compactness

The weak convergence for $P_{N,\underline{\lambda},\underline{\alpha},\underline{h}}$ also requires good convergence properties for the measure $\widehat{P}_{n,\underline{\lambda},\underline{\alpha},\underline{h}}$ as $n\to\infty$. It is sufficient that the sequence $\{\widehat{P}_{n,\underline{\lambda},\underline{\alpha},\underline{h}}\}$ be relatively compact, i.e., that every sequence contained a subsequence weakly convergent to a certain probability measure. This requirement can be replaced by a weaker one, the tightness of $\{\widehat{P}_{n,\underline{\lambda},\underline{\alpha},\underline{h}}\}$, i.e., that, for every $\varepsilon>0$, there exists a compact set $K\subset H^r(D)$, such that $\widehat{P}_{n,\underline{\lambda},\underline{\alpha},\underline{h}}(K)>1-\varepsilon$ for all $n\in\mathbb{N}$.

We will reduce the proof of tightness for $\{\widehat{P}_{n,\underline{\lambda},\underline{\alpha},\underline{h}}\}$ to that of sequences of marginal measures

$$\widehat{P}_{n,\lambda_j,\alpha_j,h_j}(A)=\widehat{P}_{n,\lambda_j,\alpha_j,h_j}\Big(\underbrace{H(D)\times\cdots\times H(D)}_{j-1}\times A$$
$$\times H(D)\times\cdots\times H(D)\Big),\quad A\in\mathcal{B}(H(D)),\ j=1,\dots,r.$$

Lemma 5. *The sequence* $\{\widehat{P}_{n,\lambda_j,\alpha_j,h_j}:n\in\mathbb{N}\}$ *is tight for all* λ_j, α_j *and* $h_j>0$, $j=1,\dots,r$.

Proof. We take arbitrary λ, α and h. The estimates (3.1) and (3.2) together with the mentioned Gallagher lemma, for $1/2<\sigma<1$, implies

$$\sum_{k=0}^{N}|L(\lambda,\alpha,\sigma+ikh|^2\ll_{\lambda,\alpha,h,\sigma}N. \tag{4.1}$$

Let K_l be a compact set from the definition of the metric ρ. Then (4.1) and the Cauchy integral formula give

$$\sum_{k=0}^{N}\sup_{s\in K_l}|L(\lambda,\alpha,s+ikh)|\ll_{l,\lambda,\alpha,h}\left(N\sum_{k=0}^{N}\sup_{s\in K_l}|L(\lambda,\alpha,s+ikh)|^2\right)^{1/2}$$
$$\ll_{l,\lambda,\alpha,h}N.$$

Hence, in view of Lemma 3, we have

$$\sup_{n\in\mathbb{N}}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\sup_{s\in K_l}|L_n(\lambda,\alpha,s+ikh)| \leqslant \sup_{n\in\mathbb{N}}\limsup_{N\to\infty}\frac{1}{N+1} \times\sum_{k=0}^{N}\sup_{s\in K_l}|L_n(\lambda,\alpha,s+ikh)| + \sup_{n\in\mathbb{N}}\limsup_{N\to\infty}\sum_{k=0}^{N}\sup_{s\in K_l}|L(\lambda,\alpha,s+ikh) - L_n(\lambda,\alpha,s+ikh)| \leqslant R_{l,\lambda,\alpha,h}<\infty. \tag{4.2}$$

Let the random variable ξ_N be defined on a certain probability space $(\widehat{\Omega},\mathcal{B}(\widehat{\Omega}), P)$ and have the distribution

$$P\{\xi_N=k\}=1/(N+1),\quad k=0,1,\dots,N.$$

On the probability space $(\widehat{\Omega},\mathcal{B}(\widehat{\Omega}),P)$, define the $H^r(D)$-valued random elements

$$X_{N,n,\underline{\lambda},\underline{\alpha},\underline{h}}(s)=(X_{N,n,\lambda_1,\alpha_1,h_1}(s),\dots,X_{N,n,\lambda_r,\alpha_r,h_r}(s))=\underline{L}_n(\underline{\lambda},\underline{\alpha},s+i\xi_N\underline{h}),$$
$$X_{n,\underline{\lambda},\underline{\alpha},\underline{h}}(s)=(X_{n,\lambda_1,\alpha_1,h_1}(s),\dots,X_{n,\lambda_r,\alpha_r,h_r}(s)),$$

which has the distribution $\widehat{P}_{n,\underline{\lambda},\underline{\alpha},\underline{h}}$. Denote by $\xrightarrow{\mathcal{D}}$ the convergence in distribution. Then in view of Lemma 2,

$$X_{N,n,\underline{\lambda},\underline{\alpha},\underline{h}}\xrightarrow[N\to\infty]{\mathcal{D}}X_{n,\underline{\lambda},\underline{\alpha},\underline{h}}. \tag{4.3}$$

From this, it follows that

$$X_{N,n,\lambda,\alpha,h}\xrightarrow[N\to\infty]{\mathcal{D}}X_{n,\lambda,\alpha,h}. \tag{4.4}$$

Let $\varepsilon>0$ be fixed, and $M_l=M_l(\lambda,\alpha,h,\varepsilon)=2^lR_{l,\lambda,\alpha,h}\varepsilon^{-1},\quad l\in\mathbb{N}$. Then, by (4.4) and (4.2),

$$P\left\{\sup_{s\in K_l}|X_{N,n,\lambda,\alpha,h}(s)|>M_l\right\}\leqslant\sup_{n\in\mathbb{N}}\limsup_{N\to\infty}P\left\{\sup_{s\in K_l}|X_{N,n,\lambda,\alpha,h}(s)|>M_l\right\} \leqslant\sup_{n\in\mathbb{N}}\limsup_{N\to\infty}\frac{1}{(N+1)M_l}\sum_{k=0}^{N}\sup_{s\in K_l}|L_n(\lambda,\alpha,s+ikh)|\leqslant\frac{\varepsilon}{2^l} \tag{4.5}$$

for all $n\in\mathbb{N}$ and $l\in\mathbb{N}$. Define the set

$$K=K_\varepsilon=\left\{g\in H(D):\sup_{s\in K_l}|g(s)|\leqslant M_l, l\in\mathbb{N}\right\},$$

which is compact in the space $H(D)$. Moreover, (4.5) shows that

$$P\left\{X_{n,\lambda,\alpha,h}\in K\right\}>1-\sum_{l=1}^{\infty}\frac{\varepsilon}{2^l}=1-\varepsilon$$

for all $n\in\mathbb{N}$. This and the definition of $X_{n,\lambda,\alpha,h}$ prove the lemma. □

A simple consequence of Lemma 5 is the following

Lemma 6. *The sequence* $\{\widehat{P}_{n,\underline{\lambda},\underline{\alpha},\underline{h}} : n \in \mathbb{N}\}$ *is tight for all* $\underline{\lambda}$, $\underline{\alpha}$ *and* $\underline{h}$.

Proof. Let $\varepsilon > 0$ be fixed. Then, in virtue of Lemma 5, there exist compact sets $K_j \in H(D)$ such that

$$\widehat{P}_{n,\lambda_j,\alpha_j,h_j}(K_j) > 1 - \varepsilon/r, \quad j = 1, \dots, r, \tag{4.6}$$

for all $n \in \mathbb{N}$. Setting $K = K_1 \times \cdots \times K_r$, we have a compact set in $H^r(D)$, and, by (4.6),

$$\widehat{P}_{n,\underline{\lambda},\underline{\alpha},\underline{h}}\left(H^r(D) \setminus K\right) = \widehat{P}_{n,\underline{\lambda},\underline{\alpha},\underline{h}}\Bigg(\bigcup_{j=1}^{r}\Bigg(\underbrace{H(D) \times \cdots \times H(D)}_{j-1}\Bigg) \times \left(H(D) \setminus K_j\right)$$

$$\times H(D) \times \cdots \times H(D)\Bigg) \leqslant \sum_{j=1}^{r} \widehat{P}_{n,\lambda_j,\underline{\alpha},h_j}\left(H(D) \setminus K_j\right) \leqslant \frac{\varepsilon r}{r} = \varepsilon,$$

for all $n \in \mathbb{N}$. Therefore, $\widehat{P}_{n,\underline{\lambda},\underline{\alpha},\underline{h}}(K) > 1 - \varepsilon$ for all $n \in \mathbb{N}$. The lemma is proved. □

Corollary 1. The sequence $\{\widehat{P}_{n,\underline{\lambda},\underline{\alpha},\underline{h}} : n \in \mathbb{N}\}$ is relatively compact.

Proof. The corollary follows from Lemma 6 and Prokhorov's theorems, see, for example, [6], Theorem 6.1, which asserts that every tight family of probability measures is relatively compact. □

5 Limit theorems

Now we are ready to obtain the weak convergence for $P_{N,\underline{\lambda},\underline{\alpha},\underline{h}}$ as $N \to \infty$.

Theorem 7. *Suppose that* $\underline{\lambda}$, $\underline{\alpha}$ *and* $\underline{h}$ *are arbitrary. Then, on* $(H^r(D), \mathcal{B}(H^r(D)))$, *there exists a probability measure* $P_{\underline{\lambda},\underline{\alpha},\underline{h}}$ *such that* $P_{N,\underline{\lambda},\underline{\alpha},\underline{h}}$ *converges weakly to* $P_{\underline{\lambda},\underline{\alpha},\underline{h}}$ *as* $N \to \infty$.

Proof. On the probability space $(H^r(D), \mathcal{B}(H^r(D), m_H)$, define one more $H^r(D)$-valued random element

$$\widehat{X}_{N,\underline{\lambda},\underline{\alpha},\underline{h}}(s) = \underline{L}(\underline{\lambda}, \underline{\alpha}, s + i\xi_N \underline{h}).$$

Since, by Corollary 1, the sequence $\{\widehat{P}_{n,\underline{\lambda},\underline{\alpha},\underline{h}}\}$ is relatively compact, there exists a subsequence $\{\widehat{P}_{n_l,\underline{\lambda},\underline{\alpha},\underline{h}}\} \subset \{\widehat{P}_{n,\underline{\lambda},\underline{\alpha},\underline{h}}\}$ and a probability measure $P_{\underline{\lambda},\underline{\alpha},\underline{h}}$ on $(H^r(D), \mathcal{B}(H^r(D)))$, such that $\widehat{P}_{n_l,\underline{\lambda},\underline{\alpha},\underline{h}}$ converges weakly to $P_{\underline{\lambda},\underline{\alpha},\underline{h}}$ as $l \to \infty$. This can be written using convergence in distribution as

$$X_{n,\underline{\lambda},\underline{\alpha},\underline{h}} \xrightarrow[l\to\infty]{\mathcal{D}} P_{\underline{\lambda},\underline{\alpha},\underline{h}}. \tag{5.1}$$

Moreover, we find, for $\varepsilon > 0$,

$$P\left\{\underline{\rho}(X_{N,n,\underline{\lambda},\underline{\alpha},\underline{h}}, \widehat{X}_{N,\underline{\lambda},\underline{\alpha},\underline{h}}) \geqslant \varepsilon\right\}$$

$$\leqslant \frac{1}{(N+1)\varepsilon} \sum_{k=0}^{N} \underline{\rho}\left(\underline{L}(\underline{\lambda}, \underline{\alpha}, s + ik\underline{h}), \underline{L}_n(\underline{\lambda}, \underline{\alpha}, s + ik\underline{h})\right),$$

thus, by Lemma 4,

$$\lim_{n\to\infty}\limsup_{N\to\infty} P\left\{\underline{\rho}(X_{N,n,\underline{\lambda},\underline{\alpha},\underline{h}}, \widehat{X}_{N,\underline{\lambda},\underline{\alpha},\underline{h}}) \geqslant \varepsilon\right\} = 0.$$

This and relations (4.3) and (5.1) show that all conditions of Theorem 4.2 from [6] are fulfilled. Therefore, we have

$$\widehat{X}_{N,\underline{\lambda},\underline{\alpha},\underline{h}} \xrightarrow[N\to\infty]{\mathcal{D}} P_{\underline{\lambda},\underline{\alpha},\underline{h}},$$

and this means that $P_{N,\underline{\lambda},\underline{\alpha},\underline{h}}$ converges weakly to the measure $P_{\underline{\lambda},\underline{\alpha},\underline{h}}$ as $N \to \infty$. □

Theorem 7 implies a limit theorem for some compositions $\Psi(\underline{L}(\underline{\lambda},\underline{\alpha},s))$. Let $\Psi : H^r(D) \to H(D)$ be a certain operator, and, for $A \in \mathcal{B}(H(D))$,

$$P_{N,\Psi,\underline{\lambda},\underline{\alpha},\underline{h}}(A) = \frac{1}{N+1}\#\left\{0 \leqslant k \leqslant N : \Psi(\underline{L}(\underline{\lambda},\underline{\alpha}, s+ik\underline{h})) \in A\right\}.$$

Theorem 8. *Suppose that Ψ is a continuous operator, and $P_{\underline{\lambda},\underline{\alpha},\underline{h}}$ is a limit measure in Theorem 7. Then, for arbitrary $\underline{\lambda}$, $\underline{\alpha}$, and $\underline{h}$, $P_{N,\Psi,\underline{\lambda},\underline{\alpha},\underline{h}}$ converges weakly to the measure $P_{\underline{\lambda},\underline{\alpha},\underline{h}}\Psi^{-1}$ as $N \to \infty$.*

Proof. From the definitions of $P_{N,\underline{\lambda},\underline{\alpha},\underline{h}}$ and $P_{N,\Psi,\underline{\lambda},\underline{\alpha},\underline{h}}$, we have

$$P_{N,\Psi,\underline{\lambda},\underline{\alpha},\underline{h}} = P_{N,\underline{\lambda},\underline{\alpha},\underline{h}}\Psi^{-1}.$$

Since Ψ is continuous, using a property of preservation of weak convergence under continuous mappings, see, Theorem 5.1 of [6], and Theorem 7, we obtain that $P_{N,\Psi,\underline{\lambda},\underline{\alpha},\underline{h}}$ converges weakly to the measure $P_{\underline{\lambda},\underline{\alpha},\underline{h}}\Psi^{-1}$ as $N \to \infty$. □

6 Proof of approximation

Let P be a probability measure on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$, and the space $\mathbb{X}$ is separable. We recall that the support of P is a minimal closed set $S_P \subset \mathbb{X}$ such that $P(S_P) = 1$. The set S_P consists of all elements $x \in \mathbb{X}$, for which arbitrary open neighbourhood G_x, the inequality $P(G_x) > 0$ holds.

Proof. (Proof of Theorem 5). *Case of lower density.* Denote by $F_{\underline{\lambda},\underline{\alpha},\underline{h}}$ the support of the measure $P_{\underline{\lambda},\underline{\alpha},\underline{h}}$ in Theorem 7. Thus, $P_{\underline{\lambda},\underline{\alpha},\underline{h}}(F_{\underline{\lambda},\underline{\alpha},\underline{h}}) = 1$. Therefore, $F_{\underline{\lambda},\underline{\alpha},\underline{h}} \neq \varnothing$ and $F_{\underline{\lambda},\underline{\alpha},\underline{h}}$ is a closed set. The set

$$G(\varepsilon) = \left\{(g_1,\dots,g_r) \in H^r(D) : \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|g_j(s) - f_j(s)| < \varepsilon\right\}$$

is an open neighbourhood of $(f_1,\dots,f_r) \in F_{\underline{\lambda},\underline{\alpha},\underline{h}}$. Hence,

$$P_{\underline{\lambda},\underline{\alpha},\underline{h}}(G(\varepsilon)) > 0. \tag{6.1}$$

Thus, Theorem7 and the equivalent of weak convergence of probability measures in terms of open sets, see, Theorem 2.1 of [6], imply

$$\liminf_{N\to\infty} P_{N,\underline{\lambda},\underline{\alpha},\underline{h}}(G(\varepsilon)) \geqslant P_{\underline{\lambda},\underline{\alpha},\underline{h}}(G(\varepsilon)) > 0.$$

This and the definitions of $P_{N,\underline{\lambda},\underline{\alpha},\underline{h}}$ and $G(\varepsilon)$ prove the first part of the theorem.

Case of density. We observe that the boundaries of the sets $G(\varepsilon)$ with different ε do not intersect. Therefore, the set $G(\varepsilon)$ is a continuity set of the measure $P_{\underline{\lambda},\underline{\alpha},\underline{h}}$ for all but at most countably many $\varepsilon > 0$. Thus, Theorem 7, and the equivalence of weak convergence of probability measures in terms of continuity sets [6] and (6.1) show that the limit

$$\lim_{N\to\infty} P_{N,\underline{\lambda},\underline{\alpha},\underline{h}}(G(\varepsilon)) = P_{\underline{\lambda},\underline{\alpha},\underline{h}}(G(\varepsilon))$$

exists and is positive for all but at most countably many $\varepsilon > 0$. This and definitions of $P_{N,\underline{\lambda},\underline{\alpha},\underline{h}}$ and $G(\varepsilon)$ prove the second part of the theorem. □

Proof. (Proof of Theorem 6). We start with the support of the measure $P_{\underline{\lambda},\underline{\alpha},\underline{h}}\Psi^{-1}$. First we will show that the preimage $\Psi^{-1}\{p\}$ of a polynomial in the condition $(\Psi^{-1}\{p\}) \cap F_{\underline{\lambda},\underline{\alpha},\underline{h}} \neq \varnothing$ can be replaced by a preimage $\Psi^{-1}(G)$ of an arbitrary open set $\varnothing \neq G \subset H(D)$. Let $g \in G$. By the Mergelyan theorem on approximation of analytic functions by polynomials, see [24], there exists a polynomial $p(s)$ such that

$$\sup_{s\in K} |g(s) - p(s)| < \delta$$

for every set $K \in \mathcal{K}$. From this and the definition of the metric ρ, it follows that $\rho(g,p) < 2\delta$. Thus, if $\delta > 0$ is sufficiently small, the polynomial $p(s) \in G$. Since $(\Psi^{-1}\{p\}) \cap F_{\underline{\lambda},\underline{\alpha},\underline{h}} \neq \varnothing$, this implies that also $(\Psi^{-1}G) \cap F_{\underline{\lambda},\underline{\alpha},\underline{h}} \neq \varnothing$.

Now, let $g \in \Psi(F_{\underline{\lambda},\underline{\alpha},\underline{h}})$ be an arbitrary element, and G its arbitrary open neighbourhood. Since Ψ is continuous, the set $\Psi^{-1}G$ is also open, and contains an element of the set $F_{\underline{\lambda},\underline{\alpha},\underline{h}}$. Therefore, by a property of the support, $P_{\underline{\lambda},\underline{\alpha},\underline{h}}(\Psi^{-1}G) > 0$. Hence,

$$P_{\underline{\lambda},\underline{\alpha},\underline{h}}\Psi^{-1}(G) = P_{\underline{\lambda},\underline{\alpha},\underline{h}}(\Psi^{-1}G) > 0.$$

Moreover,

$$P_{\underline{\lambda},\underline{\alpha},\underline{h}}\Psi^{-1}(\Psi(F_{\underline{\lambda},\underline{\alpha},\underline{h}})) = P_{\underline{\lambda},\underline{\alpha},\underline{h}}(\Psi^{-1}\Psi(F_{\underline{\lambda},\underline{\alpha},\underline{h}}) = P_{\underline{\lambda},\underline{\alpha},\underline{h}}(F_{\underline{\lambda},\underline{\alpha},\underline{h}}) = 1.$$

The latter remarks show that the support of the measure $P_{\underline{\lambda},\underline{\alpha},\underline{h}}\Psi^{-1}$ is the set $\Psi(F_{\underline{\lambda},\underline{\alpha},\underline{h}})$. From this, it follows that the proof of Theorem 6 runs in the same lines as that of Theorem 5 by using Theorem 8. □

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