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Inertial Mann-Krasnoselskii Algorithm with Self Adaptive Stepsize for Split Variational Inclusion Problem and Paramonotone Equilibria

Lateef O. Jolaoso[a,b], Akindele A. Mebawondu[a] and Oluwatosin T. Mewomo[a]

[a] *School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal*
Durban, South Africa
[b] *DSI-NRF Center of Excellence in Mathematical and Statistical Sciences (CoE-MaSS)*
Johannesburg, South Africa
E-mail(*corresp.*): mewomoo@ukzn.ac.za
E-mail: 216074984@stu.ukzn.ac.za
E-mail: 216028272@stu.ukzn.ac.za

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Abstract. In this paper, we introduce a Mann-Krasnoselskii algorithm of inertial form for approximating a common solution of Spit Variational Inclusion Problem (SVIP) and Equilibrium Problem (EP) with paramonotone bifunction in real Hilbert spaces. Motivated by the self-adaptive technique, we incorporate the inertial technique to accelerate the convergence of the proposed method. Under standard and mild assumptions such as monotonicity and lower semicontinuity of the SVIP and EP associated mappings, we establish the strong convergence of the iterative algorithm. Some applications and numerical experiments are presented to illustrate the performance and behaviour of our method as well as comparing it with some related methods in the literature. Our results improve and generalize many existing results in this direction.

Keywords: split variational inclusion, equilibrium problem, pseudomonotonicity, self adaptive stepsize.

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1 Introduction

Let H be a real Hilbert space equipped with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. A multi-valued mapping $B : H \to 2^H$ is called monotone if for all $x, y \in H$, $u \in Bx$ and $v \in By$ then $\langle x - y, u - v \rangle \geq 0$. A monotone mapping $B : H \to 2^H$ is maximal if the graph $\mathbb{G}(B) = \{(x, y) \in D(B) : y \in Bx\}$ is not properly contained in the graph of any other monotone mapping. Let $B : H \to 2^H$ be a multi-valued maximal monotone mapping. The resolvent mapping $J^B_\lambda : H \to H$ associated with B is defined by

$$J^B_\lambda(x) := (I + \lambda B)^{-1}(x), \quad \forall \, x \in H,$$

for some $\lambda > 0$, where I is the identity operator on H. We note that for all $\lambda > 0$, the resolvent operator J^B_λ is single-valued, firmly nonexpansive, see e.g [2]. In 2011, Moudafi [14] introduced the following Split Variational Inclusion Problem (shortly, SVIP): Find $x^\dagger \in H_1$ such that

$$0 \in B_1(x^\dagger) \quad \text{and} \quad 0 \in B_2(Ax^\dagger), \tag{1.1}$$

where H_1 and H_2 are real Hilbert spaces, $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are multi-valued maximal monotone operators, $A : H_1 \to H_2$ is a linear bounded operator. We denote the set of solutions of (1.1) by $SVIP(B_1, B_2)$. In 2016, Chuang [7] studied the SVIP using the following descent projection method:

Algorithm 1. Descent Projection Algorithm (DPA)

Step 0: Set $n = 1$ and choose $x_1 \in H_1$.
Step 1: Given $x_n \in H_1$, compute $\{y_n\}$ using

$$y_n = J^{B_1}_{\lambda_n}[x_n - \gamma_n A^*(I - J^{B_2}_{\lambda_n})Ax_n],$$

where $\{\lambda_n\} \subset (0, \infty)$ and $\gamma_n > 0$ satisfying

$$\gamma_n \|A^*(I - J^{B_2}_{\lambda_n})Ax_n - A^*(I - J^{B_2}_{\lambda_n})Ay_n\| \leq \delta \|x_n - y_n\|, \quad \delta \in (0, 1).$$

Step 2: If $x_n = y_n$, STOP. Otherwise continue with Step 3.
Step 3: Compute $x_{n+1} \in H_1$ using

$$x_{n+1} = J^{B_1}_{\lambda_n}(x_n - \alpha_n D(x_n, y_n)),$$

where

$$D(x_n, y_n) := x_n - y_n + \gamma_n[A^*(I - J^{B_2}_{\lambda_n})Ay_n - A^*(I - J^{B_2}_{\lambda_n})Ax_n],$$
$$\alpha_n = \langle x_n - y_n, D(x_n, y_n) \rangle / \|D(x_n, y_n)\|^2.$$

Then update $n := n + 1$ and go to Step 1.

For more details and recent results on SVIP and related optimization problems, one can refer the reader to [10, 16, 27].

The Equilibrium Problem (shortly, EP) introduced by Blum and Oetlli [5] is defined as

$$\text{find} \quad x \in C \quad \text{such that} \quad f(x,y) \geq 0, \quad \forall\, y \in C, \tag{1.2}$$

where $f : C \times C \to \mathbb{R}$ is a bifunction satisfying $f(x,x) = 0$ and C is a nonempty, closed and convex subset of H. We denote the solution of Problem (1.2) by $EP(f)$. For more details and recent results on EP and related optimization problems, see [3, 11, 18].

The authors in [28] considered the following problem:

$$\text{Find} \quad x^* \in C : f(x^*, y) \geq 0 \quad \forall\, y \in C, \quad g(Ax^*) \leq g(u) \quad \forall\, u \in H_2, \tag{1.3}$$

where g is a proper lower semicontinuous convex function on H_2. They proposed the following algorithm and proved its strong convergence to a solution of Problem (1.3).

Algorithm 2. Mann-Krasnolselskii Proximal Algorithm (MKPA)

Initialization: Take positive parameters δ, ξ and real sequences $\{a_n\}, \{\delta_n\}$, $\{\beta_n\}$, $\{\epsilon_n\}, \{\rho_n\}$ satisfying

$$0 < a < a_n < b < 1, \quad 0 < \xi < \rho_n \leq 4 - \xi,$$
$$\delta_n > \delta > 0, \beta_n > 0, \quad \epsilon_n > 0, \quad \forall\, n \in \mathbb{N},$$
$$\lim_{n\to\infty} a_n = \frac{1}{2}, \quad \sum_{n=1}^{\infty} \frac{\beta_n}{a_n} = +\infty, \quad \sum_{n=1}^{\infty} \beta_n^2 < +\infty, \quad \sum_{n=1}^{\infty} \frac{\beta_n \epsilon_n}{\delta_n} < +\infty.$$

Step 0: Choose $x_1 \in C$ and let $n = 1$.
Step n: Having $x_n \in C$, take $g_n \in \partial_2^{\epsilon_n} f(x_n, x_n)$ and define

$$\alpha_n = \frac{\beta_n}{\gamma_n} \quad \text{where} \quad \gamma_n = \max\{\delta_n, \|g_n\|\}.$$

Compute $y_n = P_C(x_n - \alpha_n g_n)$, i.e.,

$$\langle y_n - x_n + \alpha_n g_n, x - y_n \rangle \geq 0 \quad \forall\, x \in C.$$

Take

$$\mu_n = \begin{cases} 0, & \text{if } \nabla h(y_n) = 0, \\ \rho_n \frac{h(y_n)}{\|\nabla h(y_n)\|^2}, & \text{if } \nabla h(y_n) \neq 0, \end{cases}$$

and compute

$$z_n = P_C(y_n - \mu_n A^*(I - prox_{\lambda g})Ay_n),$$

where $prox_{\lambda g}(u) := argmin\Big\{g(u) + \frac{1}{\lambda}\|v - u\|^2 : v \in H_2\Big\}$, and $\nabla h(x) := A^*(I - prox_{\lambda g})Ax$. Let

$$x_{n+1} = a_n x_n + (1 - a_n) z_n.$$

Motivated by the works of Moudafi [14], Chuang [6] and Yen [28] and the current research interest in this direction, we study the following problem:

$$\text{Find} \quad x^\dagger \in C : f(x^\dagger, y) \geq 0 \quad \forall\, y \in C, \quad 0 \in B_1(x^\dagger), \quad 0 \in B_2(Ax^\dagger), \quad (1.4)$$

where $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are multi-valued maximal monotone operators, $A : H_1 \to H_2$ is a bounded operator and $f : C \times C \to \mathbb{R}$ is a bifunction satisfying $f(x, x) = 0$. We denote the set of solutions of (1.4) with Γ, i.e., $\Gamma := EP(f) \cap SVIP(B_1, B_2)$. It is easy to see that Problem (1.4) contains Problems (1.1), (1.2) and (1.3).

The inertial type algorithms which originated from the heavy ball method of the two order time dynamical system can be regarded as a means of speeding up the convergence rates of iterative schemes. Some recent results on inertial algorithms can be found in [17, 19].

In many practical problem in physical science, engineering and economics, it is important to study the problem of finding a common solution of two or more optimization problems due to its possible applications to mathematical models whose constraints can be expressed as two or more optimization problems. This happen, in particular, in the practical problems such as in signal processing, network resource allocation, image recovery, see for instance [9, 13].

Our aim in this paper is to propose an inertial Mann-Krasnoelskii algorithm which converges strongly to a common solution of split inclusion problem and equilibrium problem with paramonotone bifunction. The algorithm is designed in such a way that it stepsize is chosen self-adaptively, and its strong convergence analysis does not require a prior estimate of the norm of the bounded operator. We also present some numerical examples to illustrate the performance and efficiency of our algorithm. The result in this paper improves and generalizes many recent results on EP and SVIP in the literature.

Throughout this paper, we denote the strong convergence of the sequence $\{x_n\} \subset H$ to a point $x \in H$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$; $B^{-1}(0)$ is the null set for a maximal monotone operator $B : H \to 2^H$, i.e., $B^{-1}(0) = \{x \in H : 0 \in B(x)\}$ and $N_C(x)$ is the normal cone of the set C at a point x, i.e., $N_C(x) := \{x^* \in H : \langle x - z, x^* \rangle \geq 0, \quad \forall z \in C\}$; $Fix(T)$ denotes the set of fixed point of a mapping $T : H \to H$, i.e., $Fix(T) = \{x \in H : Tx = x\}$.

2 Preliminaries

In this section, we provide some basic definitions and results which will be needed in the sequel. We recall that the metric projection P_C from H onto a nonempty, closed and convex subset $C \subseteq H$ is defined by

$$P_C x := argmin_{y \in C} \|x - y\|^2, \quad x \in H.$$

It is well known (see [10]) that P_C is characterized by the inequality

$$\langle x - P_C x, y - P_C x \rangle \leq 0 \quad \forall y \in C. \quad (2.1)$$

The following identities are well known in Hilbert spaces: For $x, y \in H$,

$$2\langle x, y \rangle = \|x\|^2 + \|y\|^2 - \|x - y\|^2, \quad \|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle. \quad (2.2)$$

DEFINITION 1. Let $T : C \to H$ be an operator. Then T is said to be demiclosed at $y \in H$ if for any sequence $\{x_n\}$ in C such that $x_n \rightharpoonup x$ and $Tx_n \to y$ imply $Tx = y$.

Lemma 1. *[6] Let H be a real Hilbert space and $B : H \to 2^H$ be a set-valued maximal monotone operator. For each $x \in H$, $\lambda > 0$ and $J_\lambda^B(x) = (I + \lambda B)^{-1}(x)$, then*

(i) J_λ^B is single-valued and firmly nonexpansive;

(ii) $D(J_\lambda^B) = H$ and $Fix(J_\lambda^B) = \{x \in H : 0 \in B(x)\}$;

(iii) $\|x - J_\lambda^B x\| \leq \|x - J_\gamma^B\|$ for all $0 < \lambda < \gamma$, $x \in H$;

(iv) Suppose $B^{-1}(0) \neq \emptyset$. Then $\|x - J_\lambda^B x\|^2 + \|J_\lambda^B x - y^*\|^2 \leq \|x - y^*\|^2$ for each $x \in H$ and $y^* \in B^{-1}(0)$;

(v) Suppose $B^{-1}(0) \neq \emptyset$. Then $\langle x - J_\lambda^B x, J_\lambda^B x - y\rangle \geq 0$ for each $x \in H$ and $y \in B^{-1}(0)$.

Lemma 2. *(see [20], Lemma 2) Let $\{v_n\}$ and $\{\delta_n\}$ be nonnegative sequences of real numbers satisfying $v_{n+1} \leq v_n + \delta_n$ with $\sum_{n=1}^{\infty} \delta_n < +\infty$. Then, the sequence $\{v_n\}$ is convergent.*

Lemma 3. *[4] Let H be a real Hilbert space, $\{a_n\}$ be a sequence of real numbers such that $0 < a < a_n < b < 1$ for all $n \geq 1$ and $\{v_n\}, \{w_n\}$ be the sequences in H such that*

$$\limsup_{n\to\infty} \|v_n\| \leq c, \quad \limsup_{n\to\infty} \|w_n\| \leq c,$$

and for some $c > 0$,

$$\limsup_{n\to\infty} \|a_n v_n + (1 - a_n) w_n\| = c.$$

Then $\lim_{n\to\infty} \|v_n - w_n\| = 0$.

Lemma 4. *[8] Let C be a nonempty closed convex subset of a real Hilbert space H and let $T : C \to H$ be a nonexpansive mapping such that $Fix(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C and $z \in H$ such that $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$ as $n \to \infty$. Then $z \in Fix(T)$.*

3 Main results

In this section, we present our algorithm and its convergence analysis. We need the following assumptions for our result:

Assumption 1 *Let*

(A1) H_1 and H_2 are real Hilbert spaces, and $A : H_1 \to H_2$ is a bounded linear operator with adjoint $A^* : H_2 \to H_1$.

(A2) $B_1 : H_1 \to 2^{H_1}$ *and* $B_2 : H_2 \to 2^{H_2}$ *are multi-valued maximal monotone operators.*

(A3) The bifunction $f : H \times H \to \mathbb{R}$ *satisfies the following:*

(B1) For each $x \in C$, $f(x,x) = 0$ *and* $f(x,\cdot)$ *is lower semicontinuous and convex on* C;

(B2) $\partial_2^\lambda f(x,x)$ *is nonempty for any* $\lambda > 0$ *and* $x \in C$ *and is bounded on any bounded subset of* C, *where* $\partial_2^\lambda f(x,x)$ *denotes* λ*-subdifferential of the convex function* $f(x,\cdot)$ *at* x, *that is*

$$\partial_2^\lambda(x,x) := \{\eta \in H_1 : \langle \eta, y - x\rangle + f(x,x) \le f(x,y) + \lambda, \quad \forall\, y \in C\}.$$

(B3) f *is pseudo-monotone on* C *with respect to every solution of the EP, that is* $f(x,x^*) \le 0$ *for any* $x \in C$, $x^* \in EP(f)$ *and* f *satisfies the following condition, which is called the para-monotonicity properly:*

$$x^* \in EP(f),\ y \in C, \quad f(x^*,y) = f(y,x^*) = 0 \Rightarrow y \in EP(f).$$

(B4) For all $x \in C$, $f(\cdot,x)$ *is weakly upper semicontinuous on* C.

(A4) Problem (1.4) *is consistent, i.e., its solution set* Γ *is nonempty.*

Now we present an inertial Mann-Krasnolselskii algorithm with self adaptive step-size for split variational inequality problem with para-monotone equilibria.

Algorithm 3. Inertial Mann-Krasnolselskii Algorithm

Initialization: Pick $x_0, x_1 \in H_1$, $\theta \in [0,1)$, $\{\epsilon_n\} \subset [0,\infty)$, $\{r_n\}$, $\{a_n\}$, $\{\rho_n\}$, $\{\beta_n\}$, $\{\lambda_n\}$ satisfying the following conditions for each $n \in \mathbb{N}$:

$$\rho_n > \rho > 0, \quad 0 < a < a_n < b < 1, \quad \beta_n > 0, \quad r_n > 0, \quad \lambda_n \ge 0;$$
$$\sum_{n=1}^{\infty} \epsilon_n < \infty, \quad \lim_{n\to\infty} a_n = \frac{1}{2}, \quad \liminf_{n\to\infty} r_n > 0;$$
$$\sum_{n=1}^{\infty} \frac{\beta_n}{\rho_n} = +\infty, \quad \sum_{n=1}^{\infty} \beta_n^2 = +\infty, \quad \sum_{n=1}^{\infty} \frac{\beta_n \lambda_n}{\rho_n} < +\infty.$$

Step 1: Given x_{n-1} and x_n, choose α_n such that $0 < \alpha_n \le \bar{\alpha}_n$, where

$$\bar{\alpha}_n = \begin{cases} \min\left\{\theta, \epsilon_n/\|x_n - x_{n-1}\|^2\right\}, & \text{if} \quad x_n \ne x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases}$$

Set

$$w_n = x_n + \alpha_n(x_n - x_{n-1}). \tag{3.1}$$

Step 2: Compute

$$y_n = J_{r_n}^{B_1}[w_n - \xi_n A^*(I - J_{r_n}^{B_2})Aw_n], \tag{3.2}$$

where ξ_n is chosen such that

$$\xi_n = \begin{cases} \frac{2\|(I-J_{r_n}^{B_2})Aw_n\|^2}{\|A^*(I-J_{r_n}^{B_2})Aw_n\|^2}, & \text{if } J_{r_n}^{B_2}Aw_n \neq Aw_n, \\ \xi, & \text{otherwise,} \end{cases} \tag{3.3}$$

where ξ is any nonnegative value.
Step 3: Take $\eta_n \in \partial_2^{\lambda_n} f(y_n, y_n)$ and define

$$\tau_n = \frac{\beta_n}{\gamma_n}, \quad \text{where} \quad \gamma_n = \max\{\rho_n, \|\eta_n\|\}.$$

Compute

$$z_n = P_C(y_n - \tau_n\eta_n). \tag{3.4}$$

Step 4: Let

$$x_{n+1} = a_n x_n + (1-a_n)z_n.$$

The following lemma can be obtained from Lemma 3.2 of [21].

Lemma 5. *For every $n \geq 1$, the following inequalities hold:*

$$(i)\ \tau_n\|\eta_n\| \leq \beta_n, \quad (ii)\ \|z_n - y_n\| \leq \beta_n.$$

Lemma 6. *The choice of the step-size defined in (3.3) is well defined.*

Proof. Take $w \in SVIP(B_1, B_2)$, then $J_r^{B_1}w = w$ and $J_r^{B_2}Aw = Aw$. Observe that

$$\begin{aligned} \|(I - J_{r_n}^{B_2})Aw_n\|^2 &= \langle (I - J_{r_n}^{B_2})Aw_n, (I - J_{r_n}^{B_2})Aw_n\rangle \\ &= \langle (I - J_{r_n}^{B_2})Aw_n, Aw_n - Aw + J_{r_n}^{B_2}Aw - J_{r_n}^{B_2}Aw_n\rangle \\ &= \langle (I - J_{r_n}^{B_2})Aw_n, Aw_n - Aw\rangle + \langle (I - J_{r_n}^{B_2})Aw_n, J_{r_n}^{B_2}Aw - J_{r_n}^{B_2}Aw_n\rangle \\ &= \langle A^*(I - J_{r_n}^{B_2})Aw_n, w_n - w\rangle + \langle (I - J_{r_n}^{B_2})Aw_n, J_{r_n}^{B_2}Aw - J_{r_n}^{B_2}Aw_n\rangle \\ &\leq \|A^*(I - J_{r_n}^{B_2})Aw_n\| \cdot \|w_n - w\| + \|(I - J_{r_n}^{B_2})Aw_n\| \cdot \|J_{r_n}^{B_2}Aw - J_{r_n}^{B_2}Aw_n\|. \end{aligned}$$

Consequently, for $n \in \mathbb{N}$, we get $\|A^*(I - J_{r_n}^{B_2})Aw_n\| \cdot \|w_n - w\| \geq 0$ and $\|(I - J_{r_n}^{B_2})Aw_n\| \cdot \|J_{r_n}^{B_2}Aw - J_{r_n}^{B_2}Aw_n\| \geq 0$. Since $J_{r_n}^{B_2}Aw_n \neq Aw_n$, then we obtain $\|A^*(I - J_{r_n}^{B_2})Aw_n\| \cdot \|w_n - w\| > 0$ and hence $\|A^*(I - J_{r_n}^{B_2})Aw_n\| > 0$. This implies that ξ_n defined in (3.3) is well defined. □

Lemma 7. *Let $x^* \in \Gamma$, then*

$$\|y_n - x^*\|^2 \leq \|x_n - x^*\|^2 + \alpha_n c_1\|x_n - x_{n-1}\|,$$

where $c_1 = \|x_n - x^*\| + \|x_{n-1} - x^*\| + 2\|x_n - x_{n-1}\|$.

Proof. Let $x^* \in \Gamma$. Then

$$\|y_n - x^*\|^2 = \|J_{r_n}^{B_1}[w_n - \xi_n A^*(I - J_{r_n}^{B_2})Aw_n] - J_{r_n}^{B_1}x^*\|^2$$

$$\begin{aligned}
&\leq \|w_n - x^* - \xi_n A^*(I - J_{r_n}^{B_2})Aw_n\|^2 \\
&= \|w_n - x^*\|^2 - 2\xi_n\langle A^*(I - J_{r_n}^{B_2})Aw_n, w_n - x^*\rangle + \xi_n^2\|A^*(I - J_{r_n}^{B_2})Aw_n\|^2 \\
&= \|w_n - x^*\|^2 - 2\xi_n\langle (I - J_{r_n}^{B_2})Aw_n, Aw_n - Ax^*\rangle + \xi_n^2\|A^*(I - J_{r_n}^{B_2})Aw_n\|^2 \\
&\leq \|w_n - x^*\|^2 - \xi_n[2\|(I - J_{r_n}^{B_2})Aw_n\|^2 + \xi_n\|A^*(I - J_{r_n}^{B_2})Aw_n\|^2]. \quad (3.5)
\end{aligned}$$

By the choice of ξ_n, we have

$$\|y_n - x^*\|^2 \leq \|w_n - x^*\|^2. \quad (3.6)$$

Also from (3.2), we have

$$\begin{aligned}
&\|w_n - x^*\|^2 = \|x_n + \alpha_n(x_n - x_{n-1}) - x^*\|^2 = \|x_n - x^*\|^2 \\
&\quad + 2\alpha_n\langle x_n - x^*, x_n - x_{n-1}\rangle + \alpha_n^2\|x_n - x_{n-1}\|^2 = \|x_n - x^*\|^2 \\
&\quad + \alpha_n(-\|x_{n-1} - x^*\|^2 + \|x_n - x^*\|^2 + \|x_n - x_{n-1}\|^2) + \alpha_n^2\|x_n - x_{n-1}\|^2 \\
&\leq \|x_n - x^*\|^2 + \alpha_n(\|x_n - x^*\|^2 - \|x_{n-1} - x^*\|^2) + 2\alpha_n\|x_n - x_{n-1}\|^2 \\
&= \|x_n - x^*\|^2 + \alpha_n(\|x_n - x^*\| + \|x_{n-1} - x^*\|)\|x_n - x_{n-1}\| + 2\alpha_n\|x_n - x_{n-1}\|^2 \\
&= \|x_n - x^*\|^2 + \alpha_n(\|x_n - x^*\| + \|x_{n-1} - x^*\| + 2\|x_n - x_{n-1}\|)\|x_n - x_{n-1}\| \\
&\leq \|x_n - x^*\|^2 + \alpha_n c_1\|x_n - x_{n-1}\|, \quad (3.7)
\end{aligned}$$

where $c_1 = \|x_n - x^*\| + \|x_{n-1} - x^*\| + 2\|x_n - x_{n-1}\|$. From (3.6) and (3.7), we have

$$\|y_n - x^*\|^2 \leq \|x_n - x^*\|^2 + \alpha_n c_1\|x_n - x_{n-1}\|.$$

□

Lemma 8. *Let $x^* \in \Gamma$. Then for each $n \geq 1$, we have*

$$\|z_n - x^*\|^2 \leq \|w_n - x^*\|^2 + 2\tau_n f(y_n, x^*) + 2\tau_n\lambda_n + 2\beta_n^2.$$

Proof. From (2.2), we get

$$\|z_n - x^*\|^2 = \|z_n - y_n + y_n - x^*\|^2 \leq \|y_n - x^*\|^2 + 2\langle y_n - z_n, x^* - z_n\rangle. \quad (3.8)$$

From (2.1) and (3.4), we have

$$\langle z_n - y_n + \tau_n\eta_n, x - z_n\rangle \geq 0, \quad \forall\ x \in C.$$

Taking $x = x^*$, we have

$$\langle z_n - y_n + \tau_n\eta_n, x^* - z_n\rangle \geq 0 \Leftrightarrow \langle \tau_n\eta_n, x^* - z_n\rangle \geq \langle y_n - z_n, x^* - z_n\rangle.$$

Hence from (3.8), we have

$$\begin{aligned}
\|z_n - x^*\|^2 &\leq \|y_n - x^*\|^2 + 2\langle \tau_n\eta_n, x^* - z_n\rangle \\
&= \|y_n - x^*\|^2 + 2\langle \tau_n\eta_n, x^* - y_n\rangle + 2\langle \tau_n\eta_n, y_n - z_n\rangle. \quad (3.9)
\end{aligned}$$

Since $\eta_n \in \partial_2^{\lambda_n} f(y_n, y_n)$, we have

$$f(y_n, x^*)-f(y_n, y_n) \geq \langle \eta_n, x^*-y_n\rangle-\lambda_n \Leftrightarrow f(y_n, x^*)+\lambda_n \geq \langle \eta_n, x^*-y_n\rangle. \quad (3.10)$$

On the other hand, from Lemma 5 it holds that

$$\langle \tau_n \eta_n, y_n - z_n\rangle \leq \tau_n \|\eta_n\| \|y_n - z_n\| \leq \beta_n^2. \quad (3.11)$$

Combining (3.9), (3.10) and (3.11), we get

$$\|z_n - x^*\|^2 \leq \|y_n - x^*\|^2 + 2\tau_n f(y_n, x^*) + 2\tau_n \lambda_n + 2\beta_n^2,$$

which together with (3.6) yields

$$\|z_n - x^*\|^2 \leq \|w_n - x^*\|^2 + 2\tau_n f(y_n, x^*) + 2\tau_n \lambda_n + 2\beta_n^2.$$

□

We now give the convergence analysis of Algorithm 3 to solution of Problem (1.4).

Theorem 1. *Suppose Assumption 1 holds and the sequence* $\{x_n\}$ *is generated by Algorithm 3. Then, the sequence* $\{x_n\}$ *strongly converges to a solution of Problem* (1.4).

Proof. Claim 1: The sequence $\{\|x_n - x^*\|^2\}$ is convergent for all $x^* \in \varGamma$. Since $x^* \in EP(f)$, and f is pseudomonotone on C with respect to every solution of EP, we have $f(y_n, x^*) \leq 0$. By the definition of x_{n+1}, we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|a_n x_n + (1-a_n) z_n - x^*\|^2 \\ &\leq a_n \|x_n - x^*\|^2 + (1-a_n)\|z_n - x^*\|^2. \end{aligned} \quad (3.12)$$

From Lemma 8 and (3.7), we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq a_n \|x_n - x^*\|^2 + (1-a_n)[\|w_n - x^*\|^2 + 2\tau_n f(y_n, x^*) \\ &+ 2\tau_n\lambda_n + 2\beta_n^2] \leq \|x_n - x^*\|^2 + (1-a_n)\alpha_n c_1 \|x_n - x_{n-1}\| + \varLambda_n, \end{aligned} \quad (3.13)$$

where $\varLambda_n = 2(1-a_n)(\tau_n\lambda_n + \beta_n^2)$.

Since $\tau_n = \frac{\beta_n}{\gamma_n}$ with $\gamma_n = \max\{\rho_n, \|\eta_n\|\}$,

$$\sum_{n=1}^{\infty} \tau_n \lambda_n = \sum_{n=1}^{\infty} \frac{\beta_n}{\gamma_n}\lambda_n \leq \sum_{n=1}^{\infty} \frac{\beta_n}{\rho_n}\lambda_n < +\infty.$$

Note that $\sum_{n=1}^{\infty} \beta_n^2 < +\infty$ and $0 < a < a_n < b < 1$ and thus, we have

$$\sum_{n=1}^{\infty} \varLambda_n < 2(1-a)\sum_{n=1}^{\infty}(\tau_n\lambda_n + \beta_n^2) < +\infty.$$

Also, we have from (3.1) that

$$\alpha_n \|x_n - x_{n-1}\|^2 \leq \bar{\alpha}_n \|x_n - x_{n-1}\|^2 \leq \epsilon_n,$$

and therefore

$$\sum_{n=1}^{\infty} \alpha_n \|x_n - x_{n-1}\|^2 < \infty.$$

Now using Lemma 2 and (3.13), we see that $\{\|x_n - x^*\|^2\}$ is convergent for all $x^* \in \Gamma$. Hence, the sequence $\{x_n\}$ is bounded. Consequently, the sequences $\{w_n\}, \{y_n\}$ and $\{z_n\}$ are bounded.

Claim 2: $\limsup_{n\to\infty} f(y_n, x^*) = 0$ for all $x^* \in \Gamma$.
From (3.13), we see that

$$-2(1-a_n)\tau_n f(y_n, x^*) \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \Lambda_n + (1-a_n)\alpha_n c_1 \|x_n - x_{n-1}\|. \tag{3.14}$$

Summing up (3.14), we get

$$\sum_{n=1}^{\infty} -2(1-a_n)\tau_n f(y_n, x^*) < +\infty.$$

On the other hand, using Assumption (A2) and the fact that $\{x_n\}$ is bounded, we get that $\|\eta_n\|$ is bounded. Thus, there is a constant $L > \delta$ such that $\|\eta_n\| \leq L$ for every $n \geq 1$, and hence

$$\frac{\gamma_n}{\rho_n} = \max\left\{1, \frac{\|\eta_n\|}{\rho_n}\right\} \leq \frac{L}{\rho}.$$

Therefore

$$\tau_n = \frac{\beta_n}{\gamma_n} \geq \frac{\rho}{L}\frac{\beta_n}{\rho_n}.$$

Since $x^* \in \Gamma$, it follows from the pseudomonotonicity of f that $-f(y_n, x^*) \geq 0$ which together with $0 < a < a_n < b < 1$ implies

$$\sum_{n=1}^{\infty} (1-b)\frac{\beta_n}{\rho_n}[-f(y_n, x^*)] < +\infty.$$

Since $\sum_{n=1}^{\infty} \frac{\beta_n}{\rho_n} = \infty$, it implies that $\limsup_{n\to\infty} f(y_n, x^*) = 0$.

Claim 3: For any $x^* \in \Gamma$, let $\{y_{n_j}\}$ be a subsequence of $\{y_n\}$ such that

$$\limsup_{n\to\infty} f(y_n, x^*) = \lim_{j\to\infty} f(y_{n_j}, x^*)$$

and y^* be a weak cluster point of $\{y_{n_j}\}$. Then y^* belongs to $EP(f)$.
Without loss of generality, we can assume that $y_{n_j} \rightharpoonup y^*$ as $j \to \infty$. Since $f(\cdot, x^*)$ is upper semi-continuous and by Claim 2, we have

$$f(y^*, x^*) \geq \limsup_{j\to\infty} f(y_{n_j}, x^*) = 0.$$

Since $x^* \in \Gamma$ and f is pseudomonotone, we have $f(y^*, x^*) \leq 0$ and so $f(y^*, x^*) = 0$. Again, by the pseudomonotonicity of f, $f(x^*, y^*) \leq 0$ and hence $f(y^*, x^*) =$

$f(x^*, y^*) = 0$. Then, by the paramonotonicity (i.e., Assumption (A3)), we can conclude that y^* is also a solution of $EP(f)$.

Claim 4: Every weak cluster point $\bar{x}$ belongs to the solution set $SVIP(B_1, B_2)$. Let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_j} \rightharpoonup \bar{x}$. Observe that

$$\sum_{n=1}^{\infty} \|w_n - x_n\| = \sum_{n=1}^{\infty} \alpha_n \|x_n - x_{n-1}\| < \infty.$$

Hence

$$\lim_{n\to\infty} \|w_n - x_n\| = 0. \tag{3.15}$$

This implies that $w_{n_j} \rightharpoonup \bar{x}$, where $\{w_j\}$ is the subsequence of $\{w_n\}$. From (3.5) and (3.12), we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 \le \|x_n - x^*\|^2 - (1 - a_n)\xi_n[2\|(I - J_{r_n}^{B_2})Aw_n\|^2 \\ + \xi_n \|A^*(I - J_{r_n}^{B_2})Aw_n\|^2] + (1 - a_n)\alpha_n c_1 \|x_n - x_{n-1}\| + \Lambda_n, \end{aligned} \tag{3.16}$$

where c_1 and Λ_n are defined as in Lemma 7 and (3.13), respectively.
Put $\Theta_n = 2\|(I - J_{r_n}^{B_2})Aw_n\|^2 + \xi_n \|A^*(I - J_{r_n}^{B_2})Aw_n\|^2$. It follows that

$$(1-a_n)\xi_n\Theta_n \le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + (1 - a_n)\alpha_n c_1 \|x_n - x_{n-1}\| + \Lambda_n.$$

This implies that

$$(1-b)\sum_{n=1}^{\infty} \xi_n\Theta_n < \|x_0 - x^*\|^2 + (1-a)c_1 \sum_{n=1}^{\infty} \alpha_n \|x_n - x_{n-1}\| + \sum_{n=1}^{\infty} \Lambda_n < +\infty.$$

Hence

$$\lim_{n\to\infty} \xi_n\Theta_n = 0.$$

Moreover, from the choice of ξ_n, for a small $\varepsilon > 0$, we have

$$\xi_n < \frac{2\|(I - J_{r_n}^{B_2})Aw_n\|^2}{\|A^*(I - J_{r_n}^{B_2})Aw_n\|^2} - \varepsilon.$$

This implies that

$$\xi_n \|A^*(I - J_{r_n}^{B_2})Aw_n\|^2 < 2\|(I - J_{r_n}^{B_2})Aw_n\|^2 - \varepsilon \|A^*(I - J_{r_n}^{B_2})Aw_n\|^2$$

and thus

$$\varepsilon \|A^*(I - J_{r_n}^{B_2})Aw_n\|^2 < 2\|(I - J_{r_n}^{B_2})Aw_n\|^2 - \xi_n \|A^*(I - J_{r_n}^{B_2})Aw_n\|^2.$$

Hence

$$\varepsilon \|A^*(I - J_{r_n}^{B_2})Aw_n\|^2 < \Theta_n \to 0, \quad \text{as} \quad n \to \infty.$$

Therefore

$$\lim_{n\to\infty} \|A^*(I - J_{r_n}^{B_2})Aw_n\|^2 = 0. \tag{3.17}$$

Similarly from (3.16), we have

$$\lim_{n\to\infty} \|(I - J_{r_n}^{B_2})Aw_n\|^2 = 0. \tag{3.18}$$

Furthermore from (3.2) and (3.5), we have

$$\begin{aligned}\|y_n - x^*\|^2 &= \|J_{r_n}^{B_1}[w_n - \xi_n A^*(I - J_{r_n}^{B_2})Aw_n] - J_{r_n}^{B_1}x^*\|^2\\ &\le \langle y_n - x^*, w_n - \xi_n A^*(I - J_{r_n}^{B_2})Aw_n - x^*\rangle\\ &= \frac{1}{2}\Big\{\|y_n - x^*\|^2 + \|w_n - \xi_n A^*(I - J_{r_n}^{B_2})Aw_n - x^*\|^2\\ &\quad - \|(y_n - x^*) - [w_n - \xi_n A^*(I - J_{r_n}^{B_2})Aw_n - x^*]\|^2\Big\}\\ &= \frac{1}{2}\Big\{\|y_n - x^*\|^2 + \|w_n - x^*\|^2 - \|y_n - w_n + \xi_n A^*(I - J_{r_n}^{B_2})Aw_n\|^2\Big\}\\ &\le \frac{1}{2}\Big\{\|y_n - x^*\|^2 + \|w_n - x^*\|^2 - [\|y_n - w_n\|^2\\ &\quad + \xi_n^2\|A^*(I - J_{r_n}^{B_2})Aw_n\|^2 - 2\xi_n\|y_n - w_n\| \cdot \|A^*(I - J_{r_n}^{B_2})Aw_n\|]\Big\}.\end{aligned}$$

Hence

$$\begin{aligned}\|y_n - x^*\|^2 \le &\|w_n - x^*\|^2 - \|y_n - x_n\|^2\\ &+ 2\xi_n\|y_n - w_n\| \cdot \|A^*(I - J_{r_n}^{B_2})Aw_n\|.\end{aligned} \tag{3.19}$$

From (3.6), (3.15) and (3.19), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le a_n\|x_n - x^*\|^2 + (1 - a_n)\|z_n - x^*\|^2\\ &\le a_n\|x_n - x^*\|^2 + (1 - a_n)\|y_n - x^*\|^2 + \Lambda_n\\ &\le a_n\|x_n - x^*\|^2 + (1 - a_n)[\|w_n - x^*\|^2 - \|y_n - w_n\|^2 + \Lambda_n\\ &\quad + 2\xi_n\|y_n - w_n\| \cdot \|A^*(I - J_{r_n}^{B_2})Aw_n\|] + \Lambda_n\\ &\le \|x_n - x^*\|^2 - (1 - a_n)\|y_n - w_n\|^2 + (1 - a_n)\alpha_n c_1\|x_n - x_{n-1}\|\\ &\quad + 2(1 - a_n)\xi_n\|y_n - w_n\| \cdot \|A^*(I - J_{r_n}^{B_2})Aw_n\| + \Lambda_n.\end{aligned}$$

This implies that

$$\begin{aligned}(1 - a_n)\|y_n - w_n\|^2 \le &\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + (1 - a_n)\alpha_n c_1\|x_n - x_{n-1}\|\\ &+ 2(1 - a_n)\xi_n\|y_n - w_n\| \cdot \|A^*(I - J_{r_n}^{B_2})Aw_n\| + \Lambda_n.\end{aligned} \tag{3.20}$$

It follows from (3.20) that

$$\begin{aligned}(1 - b)\sum_{n=1}^{\infty}\|y_n - w_n\|^2 &< \|x_0 - x^*\|^2 + (1 - a)c_1\sum_{n=1}^{\infty}\alpha_n\|x_n - x_{n-1}\|^2\\ &+ 2(1 - a)\sum_{n=1}^{\infty}\xi_n\|y_n - w_n\|| \cdot \|A^*(I - J_{r_n}^{B_2})Aw_n\| + \sum_{n=1}^{\infty}\Lambda_n < \infty.\end{aligned}$$

Hence

$$\lim_{n\to\infty}\|y_n - w_n\| = 0. \tag{3.21}$$

From (3.15) and (3.21), we have

$$\lim_{n\to\infty}\|y_n - x_n\| \le \lim_{n\to\infty}[\|y_n - w_n\| + \|w_n - x_n\|] = 0.$$

Let $\{y_{n_j}\}$ be a subsequence of $\{y_n\}$, then $y_{n_j} \rightharpoonup \bar{x}$ as $j \to \infty$. Since $y_{n_j} = J^{B_1}_{r_{n_j}}(w_{n_j} - \xi_{n_j} A^*(I - J^{B_2}_{r_{n_j}})Aw_{n_j})$, we can write

$$\frac{(w_{n_j} - y_{n_j}) + A^*(I - J^{B_2}_{r_{n_j}})Aw_{n_j}}{r_{n_j}} \in B_1(y_{n_j}). \tag{3.22}$$

By passing to limit $j \to \infty$ in (3.22) and by taking into account (3.17) and (3.21), and the fact that the graph of a maximal monotone operator is weakly-strongly closed, we obtain $0 \in B_1(\bar{x})$. Furthermore since $\|x_{n_j} - w_{n_j}\| \to 0$ and $x_{n_j} \rightharpoonup \bar{x}$ as $j \to \infty$, then $w_{n_j} \rightharpoonup \bar{x}$. Moreover since A is a bounded operator, then it is continuous and hence $Aw_{n_j} \rightharpoonup A\bar{x}$. Again by (3.18) and the fact that the resolvent $J^{B_2}_{r_n}$ is nonexpansive and Lemma 4, we obtain $A\bar{x} \in B_2(A\bar{x})$. Hence $\bar{x} \in SVIP(B_1, B_2)$. This completes the proof of Claim 4.
Note that since $\|y_n - x_n\| \to 0$, as $n \to \infty$, it follows from Claim 3 and Claim 4 that $\bar{x} \in \Gamma$.

Claim 5: Finally, we show that $\{x_n\}$ converges strongly to $\bar{x} \in \Gamma$.

By Claim 1, we can assume that

$$\lim_{n\to\infty} \|x_n - \bar{x}\| = c < +\infty.$$

From Lemma 5(ii) and (3.6), we have

$$\begin{aligned} \|z_n - \bar{x}\| &\leq \|y_n - \bar{x}\| + \|z_n - y_n\| \leq \|w_n - \bar{x}\| + \beta_n \\ &\leq \|x_n - \bar{x}\| + |\alpha_n|\|x_n - x_{n-1}\| + \beta_n. \end{aligned}$$

This implies that

$$\limsup_{n\to\infty} \|z_n - \bar{x}\| \leq \limsup_{n\to\infty} (\|x_n - \bar{x}\| + |\alpha_n|\|x_n - x_{n-1}\| + \beta_n) = c.$$

By applying Lemma 3, with $v_n = x_n - \bar{x}$, $u_n = z_n - \bar{x}$, we obtain

$$\lim_{n\to\infty} \|z_n - x_n\| = 0.$$

Following similar argument as in the proof of Theorem 1 in [28], we see that

$$\lim_{n\to\infty} x_n = \bar{x}.$$

Hence, the sequence $\{x_n\}$ converges strongly to $\bar{x}$. This completes the proof. □

4 Applications and numerical examples

In this section, we give some applications of the main result in Section 3 to the approximation of solutions of some certain nonlinear optimization problems in real Hilbert spaces. Also, we carry out some numerical experiments to test the accuracy and efficiency of our algorithm. All computational tests are carried out using MATLAB 2019a on a 8 GB RAM personal computer.

4.1 Split Minimization Problem:

Let H_1 and H_2 be real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator. Given some proper, lower semicontinuous and convex functions $g_1 : H_1 \to \mathbb{R} \cup \{+\infty\}$ and $g_2 : H_2 \to \mathbb{R} \cup \{+\infty\}$, the Split Minimization Problem (SMP) is define as

$$\text{Find} \quad \bar{x} \in H_1 \quad \text{suh that} \quad \bar{x} \in argmin\ g_1 \quad \text{and} \quad A\bar{x} \in argmin\ g_2. \tag{4.1}$$

We denote the set of solution of the SMP (4.1) with Ω_{SMP}. The SMP was first introduced by Moudafi and Thakur [15] and has attracted lots of attention in recent years, see for instance [1, 15] and reference therein.

Recall that the subdifferential of $g_1 : H_1 \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\partial g_1(x) := \{\bar{x} \in H_1 : g_1(x) + \langle y - x, \bar{x}\rangle \le g_1(y) \quad \text{for each} \ \ y \in H_1\}$$

for each $x \in H_1$. The proximity operator with respect to g_1 is defined by

$$prox_{\lambda, g_1}(x) := argmin_{z \in H_1}\Big\{g_1(z) + \frac{1}{2\lambda}\|x - z\|^2\Big\},$$

for all $x \in H_1$ and $\lambda > 0$. It is well known that ∂g_1 is maximal monotone and

$$0 \in \partial g_1(\bar{x}) \Leftrightarrow \bar{x} = prox_{\lambda g_1}(\bar{x}).$$

By setting $B_1 = \partial g_1$ and $B_2 = \partial g_2$ in Algorithm 3, we obtain an algorithm for solving the SMP.

4.2 Split Feasibility Problem:

Let C and Q be nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2 respectively, and let $A : H_1 \to H_2$ be bounded linear operator. Recall that the Split Feasibility Problem (SFP) is defined as

$$\text{Find} \quad x^* \in C \quad \text{such that} \quad Ax^* \in Q. \tag{4.2}$$

We denote the set of solution of the SFP (4.2) by Ω_{SFP}. Several iterative methods have been introduced for solving the SFP in Hilbert and Banach spaces, see for instance [12, 22, 24, 25, 26].

It is well known that the indicator function of the convex subset C (i.e., $i_C(x) = x$ if $x \in C$ and $i_C(x) = +\infty$, if $x \notin C$,) is proper, convex and lower semicontinuous and ∂i_C is maximal monotone. Also, the resolvent operator $J_\lambda^{\partial i_C} = P_C$ for all $\lambda > 0$, where P_C is the metric projection onto C. Then Algorithm 3 clearly reduces to the algorithm for solving the SFP (4.2).

4.3 Numerical examples

Example 1. Let $H = \mathbb{R}^m$ and C be a box defined by $C = \{x \in \mathbb{R}^m : -1 \le x_i \le 1,\ i = 1, 2, \ldots, m\}$. Define the bifunction f on $C \times C$ by

$$f(x, y) = (Px + Qy + q)^T(y - x) \quad \forall\ x, y \in C,$$

where $q \in \mathbb{R}^m$ and P, Q are two matrices of order m such that Q is symmetric positive semidefinite and $Q-P$ is negative semidefinite. It is easy to check that f satisfies conditions (B1)–(B4). Precisely, in our example, we work with the Euclidean norm $\mathbb{R}^m$ (with $m = 50, 200, 500$ and 1000). The vector q is the zero vector in $\mathbb{R}^m$ and the two matrices P, Q are generated randomly such that their properties are satisfied using the *'gallery ('gcdmat',m)'* function in MATLAB. The entries of matrix $A \in \mathbb{R}^m \times \mathbb{R}^m$ are randomly generated in the interval $[0,1]$, $B_1 : \mathbb{R}^m \to 2^{\mathbb{R}^m}$, $B_2 : \mathbb{R}^m \to 2^{\mathbb{R}^m}$ are define by $B_1(x) = \{2x\}$ and $B_2(x) = \{-5x\}$. The sequences $\{\beta_n\}, \{a_n\}\{\rho_n\}, \{r_n\}, \{\epsilon_n\}, \{\lambda_n\}$ are chosen such that

$$\beta_n = \frac{5}{2n+1}, \; a_n = \frac{n-1}{2n+5}, \; r_n = \frac{1}{2}, \; \epsilon_n = \frac{1}{(n+1)^4}, \; \lambda_n = 0, \; \rho_n = 4,$$
$$\tau_n = \max\{4, \|\eta_n\|\}$$

for each $n \geq 1$. We compare the numerical results of Algorithm 3 and Algorithm 3 with $\alpha_n = 0$ choosing $m = 50, 200, 500$ and 1000. In each case, the initial vectors x_0 and x_1 are also generated using $rand(m,1)$ and the stopping criteria used in each case is $\frac{\|x_{n+1}-x_n\|}{\max\{1,\|x_n\|\}} < 10^{-6}$.

The numerical computations for Example 1 are reported in Table 1, Figure 1 and Figure 2. The horizontal and vertical axes show iteration n, as well as $error(n) := \|x_n - x_{n+1}\|$, respectively. In particular, Figure 1 shows the case where $m = 50$ and $m = 200$, while Figure 2 shows the case when $m = 500$ and $m = 1000$.

Table 1 presents the CPU times in seconds and the number of iterations for the four different cases of m.

Table 1. Computation results for Example 1.

		Algorithm 3	Algorithm 3 with $\alpha_n = 0$
$m = 50$	CPU time (sec)	1.1185	1.15799
	No. of Iter.	22	32
$m = 200$	CPU time (sec)	1.7821	2.1582
	No. of Iter.	23	33
$m = 500$	CPU time (sec)	3.4738	10.7083
	No. of Iter.	24	35
$m = 1000$	CPU time (sec)	8.2317	12.5352
	No. of Iter.	24	35

Next, we give an example in an infinite dimensional Hilbert space.

Example 2. Let $H_1 = H_2 = L_2([a,b])$ with norm $\|x\|_{L_2} = \left(\int_b^a |x(t)|dt\right)^{\frac{1}{2}}$. Define $C \subseteq H_1$ and $Q \subseteq H_2$ by $C := \{x \in L_2([a,b]) : \langle u, x\rangle \leq z\}$, where $0 \neq u \in L_2([a,b])$ and $z \in \mathbb{R}$, $Q = \{y \in L_2([a,b]) : \|y-d\|_{L_2} \leq r\}$, where $d \in L_2([a,b])$ and radius $r > 0$. The projection on C and Q are define by

$$P_C(x) = \begin{cases} \frac{z-\langle u,x\rangle}{\|u\|^2_{L_2}}u + x, & \langle u,x\rangle > z, \\ x, & \langle u,x\rangle \leq z, \end{cases} \qquad P_Q(y) = \begin{cases} d + r\frac{y-d}{\|y-d\|}, & y \notin Q, \\ y, & y \in Q. \end{cases}$$

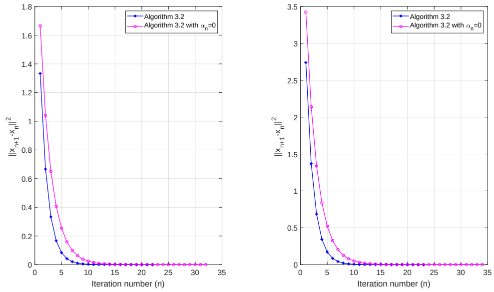

Figure 1. Example 1. Left: $m = 50$; Right: $m = 200$.

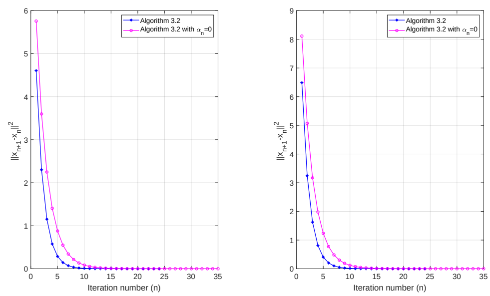

Figure 2. Example 1. Left: $m = 500$; Right: $m = 1000$.

In this example, we consider $B_1 \equiv \partial i_C$ and $B_2 \equiv \partial i_Q$, where i_C and i_Q are the indicator functions on the sets C and Q respectively. Then, the resolvent operators with respect to B_1 and B_2 are the metric projections P_C and P_Q respectively.

In particular, we choose

$$C = \{x \in L_2([0,1]) : \|x(t)\|_{L_2} \le 1\},$$
$$Q = \{x \in L_2([0,1]) : \int_0^1 |x(t) - \cos(t)|^2 dt \le 25\}.$$

Define an operator $F : C \to L^2([0,1])$ by

$$F(x)(t) = \int_0^1 (x(t) - B(t,s)p(x(s)))ds + q(t),$$

for all $x \in C$ and $t \in [0,1]$, where

$$B(t,s) = \frac{2tse^{t+s}}{e\sqrt{e^2-1}}, \quad p(x) = \cos(x), \quad q(t) = \frac{2te^t}{e\sqrt{e^2-1}}.$$

Table 2. Computation results for Example 2.

		Algorithm 3	MKPA 2	DPA 1
Case I	CPU time (sec)	1.3210	2.9709	6.1351
	No. of Iter.	17	23	40
Case II	CPU time (sec)	10. 4288	20.2761	34.9238
	No. of Iter.	21	28	48
Case III	CPU time (sec)	1.5861	2.9477	6.1550
	No. of Iter.	22	30	48
Case IV	CPU time (sec)	2.3602	9.7439	17.3865
	No. of Iter.	19	26	45

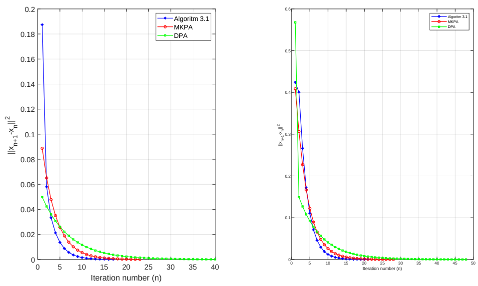

Figure 3. Example 2. Left: Case I; Right: Case II.

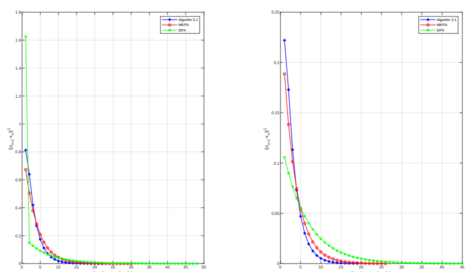

Figure 4. Example 2. Left: Case III; Right: Case IV.

As shown in [23], F is monotone and L-Lipschitz continuous with $L = 2$. Let $f\Big(x(t), y(t)\Big) = \langle Fx(t), y(t) - x(t)\rangle$, and $Ax(t) = 3x(t)$. We consider the problem

$$\text{Find } x^* \in C \text{ such that} \quad f(x^*, y) \geq 0 \quad \forall\, y \in C, \quad \text{and} \quad y^* = Ax^* \in Q. \tag{4.3}$$

Clearly, Problem (4.3) is a subclass of (1.4), hence, we can apply Algorithm 3 to solving Problem (4.3). We choose the sequences $\{a_n\}, \{\epsilon_n\}, \{\beta_n\}, \{\lambda_n\}, \{\rho_n\}$

such that

$$a_n = \frac{1}{2}, \quad \lambda_n = 0.5, \quad \beta_n = \frac{2n}{7n+3}, \epsilon_n = \frac{1}{(n+1)^2}, \quad \rho_n = 3.$$

The numerical computations for Example 2 are reported in Table 2, Figure 3 and Figure 4.

Using $\frac{\|x_{n+1}-x_n\|}{\|x_2-x_1\|} < 10^{-4}$ as stopping criterion with different choices of x_0 and x_1 given below, we compare the numerical results of Algorithm 3 with MKPA (2) and DPA (1): (i) $x_1 = t^2 - 2t + 1$ and $x_0 = 3\sin(2t)$; (ii) $x_1 = 2 - \exp(-2t)$ and $x_0 = 2t^2 - 3t$; (iii) $x_1 = \frac{3t}{4} + \frac{5t}{2} + 1$ and $x_0 = \cos(5t)$; (iv) $x_1 = \frac{12t^2}{5} - 2$ and $x_0 = \exp(-2t)/7$.

Table 2 presents the CPU times in seconds and the number of iterations for the four different cases.

Figure 3 reports Cases I and II, while Figure 4 reports Cases III and IV.

Remark 1. In conclusion, Example 1 shows that Algorithm 3 converges faster than its non-inertial type algorithm (that is, with $\alpha_n = 0$). Also from Example 2, we see that Algorithm 3 performs better than Algorithm 1 and Algorithm 2 in terms of number of iteration and cpu-time taken.

5 Conclusions

In this paper, we introduce an inertial Mann-Krasnoselskij algorithm for approximating a common solution of split variational inclusion and equilibrium problem with paramonotone bifunction. We proved a strong convergence theorem without using prior information of the norm of the bounded linear operator. More so, we provide some applications and numerical examples to illustrate the performance and applicability of the proposed method.

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