

# ON THE USE OF THE CLASSICAL NUMERICAL METHODS FOR DIFFERENTIAL DELAY EQUATIONS

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## ABSTRACT

An algorithm for the evaluation of discontinuity jumps in the DDE initial value problem is presented. That enables the use of the classical numerical methods for DDE initial value problem.

## 1. INTRODUCTION

The need for numerical solution of initial value problem for differential delay equations (DDE) arises from the fact that many biological phenomena are modelled by nonlinear equations of this type. Logistic equation [1]

$$\dot{x} = rx(1 - \frac{x_\tau}{K}) \quad (1)$$

and infectious disease model, proposed by G.I. Marchuk [2]

$$\begin{aligned} \dot{v} &= (h_1 - h_2 f)v \\ \dot{f} &= h_4(s - f) - h_8 f \cdot v \\ \dot{\xi} &= h_3 \xi(m) \cdot f_\tau \cdot v_\tau \cdot \Theta_\tau - h_5(s - 1) \\ \dot{m} &= h_6 v - h_7 m, \end{aligned} \quad (2)$$

where  $\dot{\Phi}(t) = \frac{d\Phi(t+0)}{dt}$ ,  $\Phi_\tau = \Phi(t - \tau)$ ,  $\tau$ -time delay constant,  $\Theta(t)$  - Heaviside function,  $r, K, h_i, i = 1, \dots, 8$  - model constants, are well known examples of such models.

It is one essential distinction between ODE and DDE: the solution of DDE initial value problem is *non-smooth* [3]. We will illustrate this basic property of DDE with very simple example.

EXAMPLE. Consider a initial value problem

$$\dot{x} = x_\tau, \quad \tau = 1, \tag{3}$$

$$\dot{x}(t) = 1, \quad \text{if } -\tau \leq t \leq 0, \tag{4}$$

where  $\dot{x}$  is the right derivative of  $x(t)$ .

The solution can be expressed in explicit form by using *method of steps* [3]. It is convenient to set  $x(0) = \varphi(0)$ . Thus the solution is

$$x(t) = \begin{cases} 1, & -1 \leq t < 0, \\ t + 1, & 0 \leq t < 1, \\ \frac{t^2}{2} + \frac{3}{2}, & 1 \leq t < 2, \\ \frac{t^3}{6} - \frac{t^2}{2} + 2t + \frac{1}{6}, & 2 \leq t < 3, \end{cases}$$

wherefrom we obtain

$$\begin{aligned} \Delta x_0^{(0)} &\stackrel{\text{def}}{=} x(+0) - x(-0) = 0 \\ \Delta x_0^{(1)} &\stackrel{\text{def}}{=} \dot{x}(+0) - \dot{x}(-0) = 1 \\ \Delta x_0^{(2)} &\stackrel{\text{def}}{=} \ddot{x}(+0) - \ddot{x}(-0) = 1 \\ \Delta x_0^{(3)} &\stackrel{\text{def}}{=} \dddot{x}(+0) - \dddot{x}(-0) = 2 \\ &\dots\dots\dots \end{aligned}$$

Therefore the use of classical numerical methods for DDE is troublesome. It has been shown in [3], that Runge-Kutta and Adams methods of high accuracy do not work perfectly in case (3), (4).

**2. THE SOLUTION DISCONTINUITY PROBLEM IN NUMERICAL METHODS**

More generally, the  $k$ -th derivative of the initial value problem

$$\dot{x}(t) = f(x(t), x_\tau(t), t) \tag{5}$$

$$x(t) = \varphi(t) \quad \text{for } -\tau \leq t \leq 0 \tag{6}$$

solution has finite discontinuity at point  $t_k = (k - 1)\tau$  even for smooth  $f(u, v, w)$ . Thus, the DDE solution has smoothing property: for  $t > k\tau$  its  $k$ -th derivative is continuous. Therefore, the high order accuracy numerical schemes are inapplicable in the initial part of integration interval only.

There are several ways to overcome this difficulty:

- to choose small integration step  $h$  in this initial part and any simple integration method (e.g. Euler scheme),
- to choose the integration step  $h = \frac{\tau}{p}$ , where  $p$  is positive integer and to use right derivative of the solution at its discontinuity point,
- to evaluate the derivatives discontinuity jumps in order to use this information in numerical schemes construction [4].

Consider the use of the evaluation of  $x^{(j)}(t)$  jumps at  $t = (j - 1)\tau$  in order to solve any equivalent problem with smooth solution (cf.[4]).

The following theorem is basic for further considerations.

**THEOREM [4].** *Let  $f(t), t \in (a, b)$  be a piecewise continuous function, which  $j$ -th derivatives,  $j = 0, 1, \dots, \nu$  are continuous in intervals  $(t_{k-1}, t_k), a \leq t_1 < t_2 < \dots < t_N \leq b$ , and have first kind discontinuity at points  $t = t_k$ . Denote  $\Delta f_k^{(j)}, j = 0, \dots, \nu, k = 1, \dots, N$  the discontinuity jump of the  $j$ -th derivative at  $t = t_k$ . There exist a smoothing function  $g(t)$ , for which the difference  $u(t) = f(t) - g(t)$  is  $(\nu - 1)$  - differentiable in interval  $(a, b)$ . The function  $g(t)$  can be expressed as follows:*

$$g(t) = \sum_{k=1}^N \Theta(t - t_k) \sum_{j=0}^{\nu-1} f_k^{(j)} \frac{(t - t_k)^j}{j!},$$

where

$$\Theta(t - t_k) = \begin{cases} 0, & t < t_k \\ 1, & t \geq t_k \end{cases}$$

Let us consider the initial value problem (5), (6).  $j$ -th order derivatives of its solution have first kind discontinuity jumps  $\Delta x_k^{(j)}$  at points  $t_k = k\tau$ . Evaluation of  $\Delta x_k^{(j)}$  ( $j = 1, \dots, L + 1, k = 0, \dots, L$ ) enables to construct the function  $u(t) = x(t) - g(t)$ , which is  $(L + 1)$  - differentiable and satisfies the following equation:

$$\dot{u} = f(x + g, x_\tau + g_\tau, t) - g'(t) \stackrel{\text{def}}{=} W(t, u, u_\tau) \quad (7)$$

and initial condition

$$u(t) = \varphi(t) \quad (8)$$

where

$$g'(t) = \sum_k \sum_{j=1}^{L+1} \Delta x_k^{(j)} \frac{(t - t_k)^{j-1}}{(j-1)!} \Theta(t - t_k)$$

Thus now we have a initial value problem (7), (8) with smooth solution, for which one can apply classical numeral methods.

**3. EVALUATION OF DISCONTINUITY JUMPS**

If the right-hand side of (4) is sufficiently smooth function of its arguments, we can evaluate  $\Delta_k^{(j)}$  in most elementary way, then proposed in [4]. In order to apply any classical method for the problem (5), (6) we may evaluate the quantities

$$\begin{aligned} &\Delta x_0^{(1)} \\ &\Delta x_0^{(2)}, \Delta x_1^{(2)}, \\ &\dots\dots\dots \\ &\Delta x_0^{(L+1)}, \Delta x_1^{(L+1)}, \dots, \Delta x_{L+1}^{(L+1)} \end{aligned} \tag{9}$$

Let us consider the point  $t = t_0$ . (The values  $\Delta x_k^{(j)}, k > 0$  can be find in similar way, if the solution  $x(t)$  is known in interval  $[\tau(k - 1), \tau k)$ .

For  $t \in [0, \tau)$  we have

$$\dot{u} = f(u + \sum_{j=1}^{L+1} \Delta x_0^{(j)} \frac{t^j}{j!}, \varphi(t - \tau), t) - \sum_{j=1}^{L+1} \Delta x_0^{(j)} \frac{t^{j-1}}{(j-1)!} \tag{10}$$

$u(t) = \varphi(t)$ , when  $t \in [-\tau, 0)$ .

Denote  $\partial_t^j \varphi_0 = \varphi^{(j)}(0 - 0)$ ,  $\partial_t^j \varphi_\tau = \varphi^{(j)}(-\tau + 0)$ ,  $f_1(x, x_\tau, t) = f(x, x_\tau, t)$ ,  $D^j x(t) = \frac{d^j x(t)}{dt^j}$ .

Because of the smoothness we have

$$\dot{u}(0 + 0) = \dot{u}(0 - 0) = \varphi_0^{(1)},$$

for  $t \rightarrow 0$  we obtain

$$\Delta x_0^{(1)} = f_1(\varphi_0, \varphi_\tau, 0) - \varphi_0^{(1)}.$$

Let us differtiate both sides of (5) equation according to  $t$ :

$$\begin{aligned} D^2 x &= \partial_x f_1 \cdot f_1 + \partial_{x_\tau} f_1 \cdot \partial_t \varphi_\tau + \partial_t f_1 = f_2(x, \varphi_\tau, \partial_t \varphi_\tau, t) \\ D^3 x &= \partial_x f_2 \cdot f_2 + \partial_{x_\tau} f_2 \cdot \partial_t \varphi_\tau + \partial_t f_2 \\ &\dots\dots\dots \\ D^j x &= \partial_x f_{j-1} \cdot f_{j-1} + \partial_{x_\tau} f_{j-1} \cdot \partial_t \varphi_\tau + \partial_t f_{j-1} \\ &= f_j(\varphi_0, \varphi_\tau, \dots, \partial_t \varphi_\tau, \dots, \partial_t^{(j-1)} \varphi_\tau, 0) \end{aligned} \tag{11}$$

Therefore we obtain general formula for the jumps at the point  $t = 0$ :

$$\Delta x_0^{(j)} = [f_j(\varphi_0, \varphi_\tau, \dots, \partial_t^{(j-1)} \varphi_\tau, 0) - \partial_t^j \varphi_0](j - 1)! \tag{12}$$

Denote  $x^k = x(t_k - 0)$ ,  $x_\tau^k = x(t_{k-1} + 0)$ ,  $k = 0, 1, \dots$  and formulate the general statement about jumps at the points  $t = t_k$ .

**THEOREM 2.** *If  $f(u, v, w)$  is smooth enough, then the jumps of the derivatives of the (4), (5) solution  $x(t)$  can be evaluated consequently at the points  $t_0, t_1, \dots, t_k$ ,  $k = 0, 1$  and expressed as follows:*

$$\Delta x_k^{(j)} = [f_j(x^k, x_\tau^k, \partial_t x_\tau^k, \dots, \partial_t^{j-1} x_\tau^k, t_k) - \partial_t^j x^k](j-1)! \quad (13)$$

where  $f_j$  are constructed according to (11).

**COROLLARY.** Let  $f(u, v, w)$  is smooth function of its arguments. Consider the initial value problem (6), (7). When the initial function  $\varphi(t)$  satisfies the following equalities:

$$f_j(\varphi_0, \varphi_\tau, \partial_t \varphi_\tau, \dots, \partial_t^{(j-1)} \varphi_\tau, 0) = \partial_t^j \varphi_0, \quad j = 1, \dots, N, \quad (14)$$

then the solution  $x(t)$  has continuous derivatives up to  $N$ -th order.

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