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SEMI-ANALYTICAL FINITE ELEMENTS AND THEIR APPLICATION TO MODELLING OF CYLINDRICAL SHELLS

R. Kačianauskas

1. Introduction

Despite a great difference in the existing shell models all of them provide the analytical or semi-analytical approximation of a three-dimensional body in the thickness direction with the following reduction of the dimension of the problem. As an alternative to the existing models the finite element approximation is introduced and presented in this paper. This approximation contains the essential features of semi-analytical discretisation [1]. On this base the semi-analytical finite elements (SFE) are introduced in order to develop the governing equations of linear shells. This approach has already been used for the development of the equations of beams [2-4]. Here, it is generalised and extended to more complex problems as shells [5-6].

By applying the SFE, general expressions of compatibility, equilibrium and constitutive equations of linear shell theory are derived. The equations are presented using matrix-tensor notations. The equations of cylindrical shell are considered as a particular case. By inserting the metric tensor the operators of equations are obtained explicitly.

2. Problem Formulation

Let us consider a curved shell as a three-dimensional body (Fig. 1). A curvilinear orthogonal co-ordinate system $O\hat{x}\hat{y}\hat{z}$ is chosen for the description of shell. Here and in the future the axis $O\hat{z}$ is linear. The body is bounded by two curved generatrix surfaces S_1 and S_2. The distance between two surfaces is defined as the thickness $t(\hat{x}, \hat{y})$, which is considerably smaller than the other two dimensions. The body is defined by the middle surface S equidistant from generatrix surfaces. In other words, the shell is designed moving the middle surface S along the co-ordinate $O\hat{z}$. The body is also bounded by the boundary surface $A(\hat{x}, \hat{y})$ (external cross-section of shell) normal to the middle surface.

Usually, if complex solid problems have to be considered in curvilinear co-ordinates, the tensor notations are preferable. For this reason, we shall recast the basic statements of the shell analysis and elasticity theory using tensor notations. The geometric properties of the space containing the body may be defined by tensors. In case of shell, this space and at the same time mechanical properties of the body are defined by the shape of the middle surface.

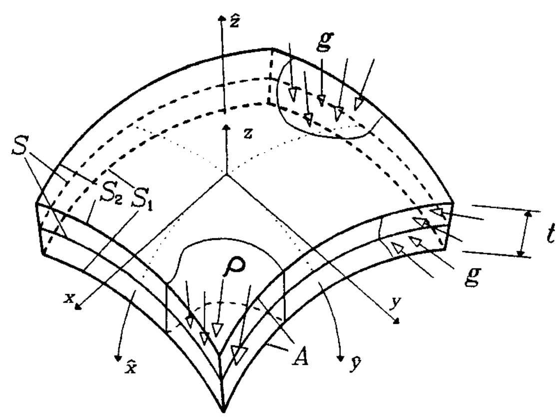

Fig. 1. Fragment of shell as three-dimensional body

The geometry of the middle surface is defined by two surface tensors. The first surface tensor (metric tensor) denoted as $\alpha(\hat{x}, \hat{y}) \equiv \alpha_{ij}(\hat{x}, \hat{y})$ describes internal (in-plane) geometry of shell and may be expressed in terms of fundamental coefficients of the first order $\hat{A}_x$ and $\hat{A}_y$

$$\alpha(\hat{x}, \hat{y}) = \begin{bmatrix} \hat{A}_x^2(\hat{x}, \hat{y}) & 0 \\ 0 & \hat{A}_y^2(\hat{x}, \hat{y}) \end{bmatrix}. \qquad (1)$$

The second surface tensor denoted as $\beta(\hat{x}, \hat{y}) \equiv \beta_{ij}(\hat{x}, \hat{y})$ describes external (out-of-plane) geometry of shell and may be expressed in terms of principal curvatures κ_x and κ_y

$$\beta(\hat{x}, \hat{y}) = \begin{bmatrix} \kappa_x(\hat{x}, \hat{y}) & 0 \\ 0 & \kappa_y(\hat{x}, \hat{y}) \end{bmatrix}. \tag{2}$$

The surface tensors mentioned above will be used for definition of the state variables and the operators of governing equations.

Now we focus our attention on mechanical variables. The classical elasticity theory is formulated in the Cartesian co-ordinates. Therefore, the classical definitions of state variables such as the stresses, the strains and the displacements and of basic operators of the equations have been referred to the Cartesian co-ordinates *Oxyz*. So, we denote the three-dimensional Cartesian variables by column vectors. This type of notations is useful in numerical analysis. The unknown stresses and strains are defined by the vector-functions $\sigma(x, y, z)$ and $\varepsilon(x, y, z)$ while displacements by $\boldsymbol{u}(x, y, z)$. The given external volume and surface loads are defined by the vector-functions $\rho(x, y, z)$ and $\boldsymbol{g}(x, y, z)$, respectively. Here, the displacements and loads contain the components of the first-rank tensors u_i, ρ_i and g_i while the stress and strain vectors contain the components of the second-rank stress tensor σ_{ij} and corresponding strain tensor ε_{ij}. For the purpose of shell analysis we define also the same variables in curvilinear co-ordinates. For instance, the stresses and strains are defined by vector-functions $\hat{\sigma}(\hat{x}, \hat{y}, \hat{z})$ and $\hat{\varepsilon}(\hat{x}, \hat{y}, \hat{z})$ while displacements and loads by vector-functions $\hat{\boldsymbol{u}}(\hat{x}, \hat{y}, \hat{z})$, $\hat{\rho}(\hat{x}, \hat{y}, \hat{z})$ and $\hat{\boldsymbol{g}}(\hat{x}, \hat{y}, \hat{z})$, respectively.

Mathematically, the curvilinear variables are physical co-ordinates of the Cartesian tensors. The relationship between both of them may be expressed using metric tensor a_{ij}. More details about application of tensors to solids and shells may be found in [7-8].

The compatibility and equilibrium operators $\hat{\mathcal{A}}^t$ and $\hat{\mathcal{A}}$ defined in curvilinear co-ordinates have principle difference in comparing with Cartesian operators. Following to differentiation rules in curvilinear co-ordinates any derivative of the tensor component is defined as covariant derivative and is expressed by using the three-index Chrystoffel symbol of the second kind Γ_{ij}^{k}.

It is assumed that physical properties are so defined that principle axis of material coincide with the curvilinear co-ordinates. In this case the Cartesian definition of constitutive operator $\hat{\mathcal{D}}(\hat{x}, \hat{y}, \hat{z}) \equiv \mathcal{D}(\hat{x}, \hat{y}, \hat{z})$ is still valid.

The boundary conditions of shell are defined on the subregions S_F and S_U of the enveloping surface. The static boundary conditions describe external surface load $\hat{\boldsymbol{g}}(\hat{x}, \hat{y}, \hat{z})\big|_{\hat{x}, \hat{y}, \hat{z} \in S_F}$ while the kinematic boundary conditions describe prescribed displacement field $\hat{\boldsymbol{u}}_P(\hat{x}, \hat{y}, \hat{z})\big|_{\hat{x}, \hat{y}, \hat{z} \in S_U}$. By assuming the external generatrix surfaces S_1 and S_2 free of the possible kinematic boundary conditions the subregions S_F and S_U are defined as follows

$$\begin{aligned} S_F(\hat{x}, \hat{y}) &= S_S(\hat{x}, \hat{y}) + A_F(\hat{x}_F, \hat{y}_F, \hat{z}_F), \\ S_U(\hat{x}, \hat{y}) &= A_U(\hat{x}_U, \hat{y}_U, \hat{z}_U), \end{aligned} \tag{3}$$

where $S_S = S_1 + S_2$.

Mixed boundary conditions at the same point $\hat{x}, \hat{y}$ are also possible.

The main issue in the development of shell theory is the separation of surface and thickness distribution of three-dimensional variables. For these purposes we define the approximation of the three-dimensional displacement field by a general expression

$$\hat{\boldsymbol{u}}(\hat{x}, \hat{y}, \hat{z}) = \boldsymbol{f}(\hat{z})\hat{\boldsymbol{U}}(\hat{x}, \hat{y}). \tag{4}$$

Here, $\boldsymbol{f}(\hat{z})$ is an approximation matrix while $\hat{\boldsymbol{U}}(\hat{x}, \hat{y})$ is the semi-discrete vector of generalised displacements. The similar approximations as presented in beam theory [2-4] may be developed for stress and strain fields.

The semi-analytical finite element method presented here is devoted for the development of the governing equations of linear shells in terms of the semi-discrete generalised variables such as displacements $\hat{\boldsymbol{U}}(\hat{x}, \hat{y})$, generalised stresses $\hat{\boldsymbol{Q}}(\hat{x}, \hat{y})$ and generalised strains $\hat{\Theta}(\hat{x}, \hat{y})$. The general semi-discrete equations derived are able to cover existing shell theories. The semi-analytical finite elements provide formal tool for the development of the higher-order theories of shells.

3. Generalisation of Concepts of the Semi-Analytical Finite Element Method and Basic Relations

The derivation of the governing equations of shells is, in fact, the partial discretisation technique, which reduces the three-dimensional solid problem by retaining the variables depending on co-ordinates $\hat{x}, \hat{y}$. We introduce to perform the partial approximation (4) in the thickness direction $\hat{z}$ by the finite element method. The finite elements describing thickness distribution are defined as semi-analytical finite elements (SFE). In case of shell we deal with the one-dimensional SFE. There exists a large number of conventional finite elements that may be applied for construction of the SFE of shells by using well-known interpolation polynomials and other approximation technique.

Let us consider the thickness of shell discretised by one-dimensional elements. The displacement field is particularly approximated by the semi-discrete expression (4) while the approximation matrix $f(\hat{z})$ is formed by conventional shape functions. The vector of the nodal displacements $\hat{U}(\hat{x}, \hat{y})$ plays the role of generalised variables. Now, the three-dimensional strain field $\hat{\varepsilon}(\hat{x}, \hat{y}, \hat{z})$ may be approximated by the expression used in the displacement approach

$$\hat{\varepsilon}(\hat{x}, \hat{y}, \hat{z}) = F(\hat{z})\hat{\Theta}(\hat{x}, \hat{y}). \qquad (5)$$

Here, the strain approximation matrix is defined by a general expression

$$F(\hat{z}) = \hat{\mathcal{A}}^{t}(\hat{x}, \hat{y}, \hat{z})f(\hat{z}), \qquad (6)$$

where compatibility operator $\hat{\mathcal{A}}^{t}(\hat{x}, \hat{y}, \hat{z})$ is defined using covariant differentiation rules. For the purpose of the semi-analytical representation this operator is split into two parts

$$\hat{\mathcal{A}}^{t} = \hat{\mathcal{A}}^{t}_{xy} + \hat{\mathcal{A}}^{t}_{z}, \qquad (7)$$

where $\hat{\mathcal{A}}^{t}_{xy}$ contains derivatives with respect to co-ordinates $\hat{x}, \hat{y}$ while $\hat{\mathcal{A}}^{t}_{z}$ contains the derivatives with respect to $\hat{z}$ and algebraic terms. As a result of decomposition (7) the vector of the generalised strains and the approximation matrix also may be decomposed into two parts:

$$\hat{\Theta}(\hat{x}, \hat{y}) \equiv \left\{\hat{\Theta}_1(\hat{x}, \hat{y}), \hat{\Theta}_2(\hat{x}, \hat{y})\right\}^{t} \text{ and}$$

$$F(\hat{z}) = \begin{bmatrix} F_1(\hat{z}) & O \\ O & F_2(\hat{z}) \end{bmatrix}.$$

The particular submatrices are

$$F_1(\hat{z}) = \hat{\mathcal{A}}^{t}_{z}(\hat{x}, \hat{y}, \hat{z})f(\hat{z}),$$
$$F_2(\hat{z}) = \Gamma_f f(\hat{z}), \qquad (8)$$

where Γ_f is the Boolean matrix reflecting the properties of the operator $\hat{\mathcal{A}}^{t}_{xy}$.

The thickness distribution of the stresses may be approximated in the same manner as (4)

$$\hat{\sigma}(\hat{x}, \hat{y}, \hat{z}) = \Phi(\hat{z})\hat{S}(\hat{x}, \hat{y}). \qquad (9)$$

Here, $\hat{S}(\hat{x}, \hat{y})$ is the vector of nodal stresses while $\Phi(\hat{z})$ is the stress approximation matrix.

The explicit evaluation of the expressions (4), (5) and (9) depends on choosing the appropriate shape functions. Once these are obtained the derivation of the governing equations follows a standard well-defined path. The development of large number of the different theories is possible by specifying the shape functions only.

The final set of the governing equations of shell is formulated in the same way as equations for three-dimensional body. It consists of the compatibility, equilibrium and constitutive relations and boundary conditions as well.

The compatibility (kinematic) equations relating the generalised strains and displacements are found by simple comparing the displacement and the strain approximations (4) and (5), respectively, and are written down as

$$\mathcal{B}_c^{\,t}(\hat{x}, \hat{y})\hat{U}(\hat{x}, \hat{y}) = \hat{\Theta}(\hat{x}, \hat{y}). \qquad (10)$$

From definition of the generalised strains (6-8) it follows that compatibility operator $\mathcal{B}_c^{\,t}(\hat{x}, \hat{y})$ naturally consists of two parts

$$\mathcal{B}^{t}_{c}(\hat{x}, \hat{y}) = \begin{bmatrix} \Gamma^{t}_{c}(\hat{x}, \hat{y}) & O \\ O & \mathcal{B}^{t}_{cx}(\hat{x}, \hat{y}) \end{bmatrix},$$

where the algebraic submatrix $\Gamma^{t}_{c}(\hat{x}, \hat{y})$ and the differential operator $\mathcal{B}^{t}_{cx}(\hat{x}, \hat{y})$ are constructed by using logical rules.

The remainder relations are derived as Euler equations associated with the modified Hellinger-Reissner functional written for the three-dimensional solid body. They containin two independent fields the stresses $\hat{\sigma}(\hat{x}, \hat{y}, \hat{z})$ and displacements $\hat{\boldsymbol{u}}(\hat{x}, \hat{y}, \hat{z})$.

According to [9] and having in mind previous notations the modified Hellinger-Reissner functional for a three-dimensional body is expressed in the following form:

$$\Pi_R(\hat{\sigma}(\hat{x}, \hat{y}, \hat{z}), \hat{\boldsymbol{u}}(\hat{x}, \hat{y}, \hat{z})) = \\ = \int_V \hat{\sigma}^t(\hat{x}, \hat{y}, \hat{z}) \hat{\mathcal{A}}^t(\hat{x}, \hat{y}, \hat{z}) \hat{\boldsymbol{u}}(\hat{x}, \hat{y}, \hat{z}) \mathrm{d}V - \\ - \frac{1}{2} \int_V \hat{\sigma}^t(\hat{x}, \hat{y}, \hat{z}) \mathcal{D}(\hat{x}, \hat{y}, \hat{z}) \hat{\sigma}(\hat{x}, \hat{y}, \hat{z})) \mathrm{d}V - \\ - \int_V \hat{\boldsymbol{u}}^t(\hat{x}, \hat{y}, \hat{z}) \hat{\rho}(\hat{x}, \hat{y}, \hat{z}) \mathrm{d}V - \\ - \int_{S_F} \hat{\boldsymbol{u}}^t(\hat{x}, \hat{y}, \hat{z}) \hat{\boldsymbol{g}}(\hat{x}, \hat{y}, \hat{z}) \mathrm{d}S - \\ - \int_{S_U} (\hat{\boldsymbol{u}}(\hat{x}, \hat{y}, \hat{z}) - \hat{\boldsymbol{u}}_P(\hat{x}, \hat{y}, \hat{z}))^t \hat{\mathcal{A}}_n(\hat{x}, \hat{y}, \hat{z}) \hat{\sigma}(\hat{x}, \hat{y}, \hat{z}) \mathrm{d}S. \tag{11}$$

where the matrix $\hat{\mathcal{A}}_n(\hat{x}, \hat{y}, \hat{z})$ is an algebraic operator of the static boundary conditions while $\hat{\boldsymbol{u}}_P(\hat{x}, \hat{y}, \hat{z})$ is the vector of the prescribed displacements.

Following standard arguments in the calculus of variations [9] and applying the Green's theorem a variation of the functional (11) is adopted for a shell by taking into account the definition of the enveloping surface (3).

After certain manipulations it is expressed as

$$\delta\Pi_R(\hat{\sigma}(\hat{x}, \hat{y}, \hat{z}), \hat{\boldsymbol{u}}(\hat{x}, \hat{y}, \hat{z})) = \\ = \int_V (\delta\hat{\sigma}(\hat{x}, \hat{y}, \hat{z}))^t \hat{\mathcal{A}}^t(\hat{x}, \hat{y}, \hat{z}) \hat{\boldsymbol{u}}(\hat{x}, \hat{y}, \hat{z}) \mathrm{d}V - \\ - \int_V (\delta\hat{\sigma}(\hat{x}, \hat{y}, \hat{z}))^t \mathcal{D}(\hat{x}, \hat{y}, \hat{z}) \hat{\sigma}(\hat{x}, \hat{y}, \hat{z})) \mathrm{d}V - \\ - \int_V (\delta\hat{\boldsymbol{u}}(\hat{x}, \hat{y}, \hat{z}))^t \hat{\mathcal{A}}(\hat{x}, \hat{y}, \hat{z}) \hat{\sigma}(\hat{x}, \hat{y}, \hat{z}) \mathrm{d}V - \\ - \int_V (\delta\hat{\boldsymbol{u}}(\hat{x}, \hat{y}, \hat{z}))^t \hat{\rho}(\hat{x}, \hat{y}, \hat{z}) \mathrm{d}V + \\ + \int_{S_F} (\delta\hat{\boldsymbol{u}}(\hat{x}, \hat{y}, \hat{z}))^t \hat{\mathcal{A}}_{nz}(\hat{x}, \hat{y}, \hat{z}) \hat{\sigma}(\hat{x}, \hat{y}, \hat{z}) \mathrm{d}S - \\ - \int_{S_F} (\delta\hat{\boldsymbol{u}}(\hat{x}, \hat{y}, \hat{z}))^t \hat{\boldsymbol{g}}(\hat{x}, \hat{y}, \hat{z}) \mathrm{d}S + \\ + \int_{A_F} (\delta\hat{\boldsymbol{u}}(\hat{x}, \hat{y}, \hat{z}))^t \hat{\mathcal{A}}_{nxy}(\hat{x}, \hat{y}) \hat{\sigma}(\hat{x}, \hat{y}, \hat{z}) \mathrm{d}A - \\ - \int_{A_F} (\delta\hat{\boldsymbol{u}}(\hat{x}, \hat{y}, \hat{z}))^t \hat{\boldsymbol{g}}(\hat{x}, \hat{y}, \hat{z}) \mathrm{d}A - \\ - \int_{A_U} (\delta\hat{\sigma}(\hat{x}, \hat{y}, \hat{z}))^t \hat{\mathcal{A}}^t_{nxy}(\hat{x}, \hat{y})(\hat{\boldsymbol{u}}(\hat{x}, \hat{y}, \hat{z}) - \hat{\boldsymbol{u}}_P(\hat{x}, \hat{y}, \hat{z})) \mathrm{d}A.$$

Taking into account the approximations (4), (5) and (9) the semi-discrete expression of variation of the functional (11) now can be rewritten for shell as

$$\delta\Pi_R\left(\hat{\boldsymbol{S}}(\hat{x}, \hat{y}), \hat{\boldsymbol{U}}(\hat{x}, \hat{y})\right) = \\ = \int_V \left(\delta\hat{\boldsymbol{S}}(\hat{x}, \hat{y})\right)^t \boldsymbol{\Phi}^t(\hat{z}) \boldsymbol{F}(\hat{z}) \hat{\Theta}(\hat{x}, \hat{y}) \mathrm{d}V - \\ - \int_V \left(\delta\hat{\boldsymbol{S}}(\hat{x}, \hat{y})\right)^t \boldsymbol{\Phi}^t(\hat{z}) \mathcal{D}(\hat{x}, \hat{y}, \hat{z}) \boldsymbol{\Phi}(\hat{z}) \hat{\boldsymbol{S}}(\hat{x}, \hat{y}) \mathrm{d}V + \\ + \int_V \left(\delta\hat{\boldsymbol{U}}(\hat{x}, \hat{y})\right)^t \boldsymbol{f}^t(\hat{z}) \hat{\mathcal{A}}(\hat{x}, \hat{y}, \hat{z}) \boldsymbol{\Phi}(\hat{z}) \hat{\boldsymbol{S}}(\hat{x}, \hat{y}) \mathrm{d}V - \\ - \int_V \left(\delta\hat{\boldsymbol{U}}(\hat{x}, \hat{y})\right)^t \boldsymbol{f}^t(\hat{z}) \hat{\rho}(\hat{x}, \hat{y}, \hat{z}) \mathrm{d}V + \\ + \int_S \left(\delta\hat{\boldsymbol{U}}(\hat{x}, \hat{y})\right)^t \boldsymbol{f}^t(\hat{z}) \hat{\mathcal{A}}_{nz}(\hat{x}, \hat{y}, \hat{z}) \boldsymbol{\Phi}(\hat{z}) \hat{\boldsymbol{S}}(\hat{x}, \hat{y}) \mathrm{d}S - \\ - \int_{S_F(\hat{x}, \hat{y})} \left(\delta\hat{\boldsymbol{U}}(\hat{x}, \hat{y})\right)^t \boldsymbol{f}^t(\hat{z}) \hat{\boldsymbol{g}}(\hat{x}, \hat{y}, \hat{z}) \mathrm{d}S + \\ + \int_{A_F(\hat{x}_F, \hat{y}_F)} \left(\delta\hat{\boldsymbol{U}}(\hat{x}, \hat{y})\right)^t \boldsymbol{f}^t(\hat{z}) \hat{\mathcal{A}}_{nxy}(\hat{x}, \hat{y}) \boldsymbol{\Phi}(\hat{z}) \mathrm{d}A - \\ - \int_{A_F(\hat{x}_F, \hat{y}_F)} \left(\delta\hat{\boldsymbol{U}}(\hat{x}, \hat{y})\right)^t \boldsymbol{f}^t(\hat{z}) \hat{\boldsymbol{g}}(\hat{x}, \hat{y}, \hat{z}) \mathrm{d}A - \\ - \int_{A_U(\hat{x}_U, \hat{y}_U)} \left(\delta\hat{\boldsymbol{S}}(\hat{x}, \hat{y})\right)^t \boldsymbol{\Phi}^t(\hat{z})^t \cdot \\ \cdot \hat{\mathcal{A}}^t_{nx}(\hat{x}, \hat{y}) \boldsymbol{f}(\hat{z}) \left(\hat{\boldsymbol{U}}(\hat{x}, \hat{y}) - \hat{\boldsymbol{U}}_P(\hat{x}, \hat{y})\right) \mathrm{d}A. \tag{12}$$

The operator $\hat{\mathcal{A}}_{nz}(\hat{x}, \hat{y})$ describes the boundary conditions on generatrix surface while $\hat{\mathcal{A}}_{nxy}(\hat{x}, \hat{y})$ the boundary conditions on external cross-section.

In order to modify the three-dimensional expression (12) to a two-dimensional one, the distributed fields of the external loads can be described by their resultants $\boldsymbol{p}_V$, $\boldsymbol{p}_S$ and $\boldsymbol{p}_A$. The other integrals in the variation (12) may be simplified in the same manner and new equilibrium operators $\mathcal{B}_{es}$ are introduced. Having in mind all of the previous notations (3-9) the total variation of Hellinger-Reissner functional may be expressed in terms of two-dimensional integrals

$$\delta\Pi_R\left(\boldsymbol{S}(\hat{x}, \hat{y}), \hat{\boldsymbol{U}}(\hat{x}, \hat{y})\right) = \\ = \int_S \left(\delta\hat{\boldsymbol{S}}(\hat{x}, \hat{y})\right)^t \left(\boldsymbol{C}^t(\hat{x}, \hat{y}) \hat{\Theta}(\hat{x}, \hat{y}) - \boldsymbol{D}_s(\hat{x}, \hat{y}) \hat{\boldsymbol{S}}(\hat{x}, \hat{y})\right) \mathrm{d}S + \\ + \int_S \left(\delta\hat{\boldsymbol{U}}(\hat{x}, \hat{y})\right)^t \left(\mathcal{B}_{es}(\hat{x}, \hat{y}) \hat{\boldsymbol{S}}(\hat{x}, \hat{y}) - \boldsymbol{p}_V(\hat{x}, \hat{y})\right) \mathrm{d}S + \\ + \int_S \left(\delta\hat{\boldsymbol{U}}(\hat{x}, \hat{y})\right)^t \left(\mathcal{B}_{esnz}(\hat{x}, \hat{y}) \hat{\boldsymbol{S}}(\hat{x}, \hat{y}) - \boldsymbol{p}_S(\hat{x}, \hat{y})\right) \mathrm{d}S +$$

$$+ \left(\delta\hat{U}(\hat{x},\hat{y})\right)^{t}\left(\mathcal{B}_{esn\,xy}(\hat{x},\hat{y})\hat{S}(\hat{x},\hat{y}) - p_A(\hat{x},\hat{y})\right)\Big|_{\hat{x},\hat{y}\in S_F} -$$
$$- \left(\delta\hat{S}(\hat{x},\hat{y})\right)^{t}\mathcal{B}_{esn\,xy}^{t}(\hat{x},\hat{y})\left(\hat{U}(\hat{x},\hat{y}) - \hat{U}_P(\hat{x},\hat{y})\right)\Big|_{\hat{x},\hat{y}\in S_U}. \tag{13}$$

Here, D_s is constitutive operator. The introduced matrix

$$C(\hat{x},\hat{y}) = \int_{t(\hat{x},\hat{y})} F^{t}(\hat{z})\Phi(\hat{z})\,\mathrm{d}t \tag{14}$$

may be considered as an approximation matrix relating stresses $\hat{S}(\hat{x},\hat{y})$ with their resultants - generalised stresses $\hat{Q}(\hat{x},\hat{y})$

$$\hat{Q}(\hat{x},\hat{y}) = C(\hat{x},\hat{y})\hat{S}(\hat{x},\hat{y}). \tag{15}$$

The stationarity condition of the functional Π_R

$$\delta\Pi_R = 0$$

provides independent variations of $\hat{S}(\hat{x},\hat{y})$ and $\hat{U}(\hat{x},\hat{y})$ and lead to a set of equations

$$\begin{Bmatrix} \dfrac{\partial\Pi_R}{\partial\hat{S}} \\ \dfrac{\partial\Pi_R}{\partial\hat{U}} \end{Bmatrix} = O. \tag{16}$$

Adopting condition (16) for the functional (13) provides the Euler equations. The equations (16) together with the compatibility relation (10) form a set of the governing equations of the shell theory expressed in terms of the generalised displacements $\hat{U}(\hat{x},\hat{y})$, the generalised strains $\hat{\Theta}(\hat{x},\hat{y})$ and the three-dimensional stresses $\hat{S}(\hat{x},\hat{y})$.

The generalisation of them and complete transformation of the stresses $\hat{S}(\hat{x},\hat{y})$ to the generalised stresses $\hat{Q}(\hat{x},\hat{y})$ according to (15) needs additional considerations. It depends on the type of the semi-analytical element: isoparametric, subparametric or superparametric. This transformation technique is already defined for the semi-analytical elements of beams [3]. The main difficulties occur, when the rank of transformation matrix $C(\hat{x},\hat{y})$ differs from the number of components of both vectors $\hat{S}(\hat{x},\hat{y})$ and $\hat{Q}(\hat{x},\hat{y})$. In spite of appropriate differences it is possible to modify the relationship (15) expressing them in terms of modified model variables $\hat{S}_M(\hat{x},\hat{y})$ and $\hat{Q}_M(\hat{x},\hat{y})$. In this case, for the forward transformation

$$\hat{Q}_M(\hat{x},\hat{y}) = C_M(\hat{x},\hat{y})\hat{S}_M(\hat{x},\hat{y}) \tag{17}$$

the corresponding backward transformation may be always uniquely established. Additional transformations relating initial and modified variables have to be also obtained. After performing some matrix manipulations, finally, the set of governing equations contains

compatibility equations

$$\mathcal{B}_{cM}^{t}(\hat{x},\hat{y})\hat{U}_M(\hat{x},\hat{y}) - \hat{\Theta}_M(\hat{x},\hat{y}) = O \in S; \tag{18a}$$

equilibrium equations

$$\mathcal{B}_{eM}(\hat{x},\hat{y})\hat{Q}_M(\hat{x},\hat{y}) - p_V(\hat{x},\hat{y}) = O \in S; \tag{18b}$$

constitutive equations

$$\mathcal{D}_M(\hat{x},\hat{y})\hat{Q}_M(\hat{x},\hat{y}) - \hat{\Theta}_M(\hat{x},\hat{y}) = O \in S; \tag{18c}$$

static boundary conditions

$$\mathcal{B}_{eMn\,z}(\hat{x},\hat{y})\hat{Q}_M(\hat{x},\hat{y}) - p_S(\hat{x},\hat{y}) = O \in S; \tag{18c}$$

$$\mathcal{B}_{eMn\,xy}(\hat{x},\hat{y})\hat{Q}_M(\hat{x},\hat{y}) - p_A(\hat{x},\hat{y}) = O \in A_F; \tag{18d}$$

kinematic boundary conditions

$$\hat{U}_M(\hat{x},\hat{y}) - \hat{U}_{PM}(\hat{x},\hat{y}) = O \quad \in A_U. \tag{18e}$$

The solution of the set of governing equations (18) provides the distribution of modified generalised variables. In some of the cases, this solution is insufficient for complete recovering of the three-dimensional stress-strain fields in the shell because some of the initial generalised variables are lost by the modification (15). This situation is well-known in classical structural mechanics and described in terms of statically (kinematically) undeterminate systems where redundant variables have to be found from additional equations.

The basic relations (18) proposed have no preliminary limitation due to geometry of shell or due to approximation laws. The semi-analytical finite element approach is the formal possibility to develop the higher-order shell theories with the desired degree of accuracy. As usual, the plates and beams may be considered as a particular case of shell.

4. Modelling Cylindrical Shells

In order to illustrate the use of the semi-analytical finite elements in the shell theory, we will deal with a circular cylindrical shell. The shell is considered as a particular case of rotational shells. The necessary tensor characteristics of shell are presented in [7] while the geometry is illustrated in Fig. 2. The global geometry of the middle surface is defined in cylindrical co-ordinates $Or\varphi z$. The orthogonal surface co-ordinates $O\hat{x}\hat{y}\hat{z}$ will be used for definition of the state variables. The single geometric parameter R defines the geometry for modelling needs. The constant thickness $t(\varphi, z) \equiv t$ and constant elasticity matrix $\mathcal{D}(r, \varphi, z) \equiv \mathcal{D}$ are assumed usually. The first surface tensor is defined by (1), where fundamental coefficients $\hat{A}_x(\hat{x}, \hat{y}) \equiv \hat{A}_x$ and $\hat{A}_y(\hat{x}, \hat{y}) \equiv \hat{A}_y$ are

$$\hat{A}_x = 1 \text{ and } \hat{A}_y = R. \qquad (19a)$$

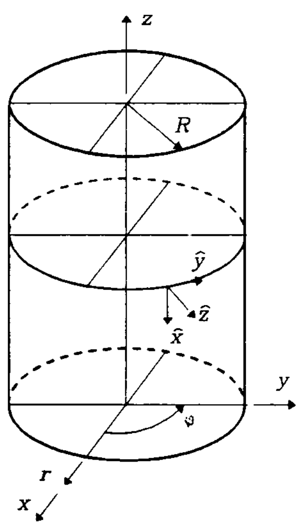

Fig. 2. Geometry and co-ordinates of cylindrical shell

The second surface tensor $\beta(\hat{x}, \hat{y}) \equiv \beta$ in (2) now is determinated as

$$\beta(\hat{x}, \hat{y}) = \begin{bmatrix} 0 & 0 \\ 0 & -\dfrac{1}{R} \end{bmatrix}. \qquad (19b)$$

Three-dimensional variables are defined by the vector-functions of displacements, strains and stresses:

$\hat{u}(\hat{x}, \hat{y}, \hat{z}) \equiv \{\hat{u}_x(\hat{x}, \hat{y}, \hat{z}), \hat{u}_y(\hat{x}, \hat{y}, \hat{z}), \hat{u}_z(\hat{x}, \hat{y}, \hat{z})\}^t$,
$\hat{\varepsilon}(\hat{x}, \hat{y}, \hat{z}) \equiv \{\hat{\varepsilon}_{xx}(\hat{x}, \hat{y}, \hat{z}), \hat{\varepsilon}_{yy}(\hat{x}, \hat{y}, \hat{z}), \hat{\gamma}_{xy}(\hat{x}, \hat{y}, \hat{z}),$
$\hat{\gamma}_{xz}(\hat{x}, \hat{y}, \hat{z}), \hat{\gamma}_{yz}(\hat{x}, \hat{y}, \hat{z})\}^t$, $\hat{\sigma}(\hat{x}, \hat{y}, \hat{z}) \equiv \{\hat{\sigma}_{xx}(\hat{x}, \hat{y}, \hat{z}),$
$\hat{\sigma}_{yy}(\hat{x}, \hat{y}, \hat{z}), \hat{\tau}_{xy}(\hat{x}, \hat{y}, \hat{z}), \hat{\tau}_{xz}(\hat{x}, \hat{y}, \hat{z}), \hat{\tau}_{yz}(\hat{x}, \hat{y}, \hat{z})\}^t$.

Thus, the pinching of normal is neglected while $\hat{\sigma}_{zz}(\hat{x}, \hat{y}, \hat{z}) \equiv 0$. Physical co-ordinates of the tensor variables are expressed as usual ([7]), where fundamental coefficients are taken from (19).

The Christoffel symbols defining the internal properties of the middle surface finally provide zero values for selected entries.

They are

$$\Gamma^1_{11} = \Gamma^2_{11} = \Gamma^1_{22} = \Gamma^2_{22} = \Gamma^1_{12} = \Gamma^2_{12} = \Gamma^1_{21} = \Gamma^2_{21} = 0. \qquad (20a)$$

The Christoffel symbols defining influence of curvature for in-surface variables are

$$\Gamma^3_{11} = \Gamma^3_{12} = 0; \; \Gamma^3_{22} = -\frac{1}{R} \qquad (20b)$$

while the non-zero Christoffel symbols defining influence of curvature for transversal variables are

$$\Gamma^2_{23} = \frac{1}{R}. \qquad (20c)$$

The three-dimensional relations are valid for any generatrix surface, thus, in general, $R \equiv R(\hat{z})$. To simplify final relation it is assumed that $t/R << 1$ which finally let us to change the curvature of generatrix surface by the curvature of middle surface. Thus, $R(\hat{z}) \approx R(0) = const$.

By selecting the necessary components of Christoffel symbols (20) as well as corresponding metric tensor decomposition (7) applied for three-dimensional compatibility operator is expressed as follows

$$\mathcal{A}^t = \begin{bmatrix} \dfrac{\partial}{\partial \hat{x}} & 0 & 0 \\ 0 & \dfrac{1}{R}\dfrac{\partial}{\partial \hat{y}} & 0 \\ \dfrac{1}{R}\dfrac{\partial}{\partial \hat{y}} & \dfrac{\partial}{\partial \hat{x}} & 0 \\ 0 & 0 & \dfrac{\partial}{\partial \hat{x}} \\ 0 & 0 & \dfrac{1}{R}\dfrac{\partial}{\partial \hat{y}} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \dfrac{1}{R} \\ 0 & 0 & 0 \\ \dfrac{\partial}{\partial \hat{z}} & 0 & 0 \\ 0 & \dfrac{\partial}{\partial \hat{z}} - \dfrac{1}{R} & 0 \end{bmatrix}.$$

Now we return to the semi-analytical elements. The thickness of shell is discretised by single one-dimensional element (Fig. 3a). The element is defined by co-ordinate $\hat{z}$ of two nodal points 1 and 2. The linear distribution of surface (in-plane) displacement components $\hat{u}_x(\hat{z})$ as well as $\hat{u}_y(\hat{z})$ is assumed while $\hat{u}_z(\hat{z})$ is taken as constant one (Fig. 3b). The vector of generalised displacements of shell is defined as $\hat{U}(\hat{x}, \hat{y}) \equiv \{\hat{U}_{x1}(\hat{x}, \hat{y}), \hat{U}_{x2}(\hat{x}, \hat{y}), \hat{U}_{y1}(\hat{x}, \hat{y}), \hat{U}_{y2}(\hat{x}, \hat{y}), \hat{U}_z(\hat{x}, \hat{y})\}^t$.

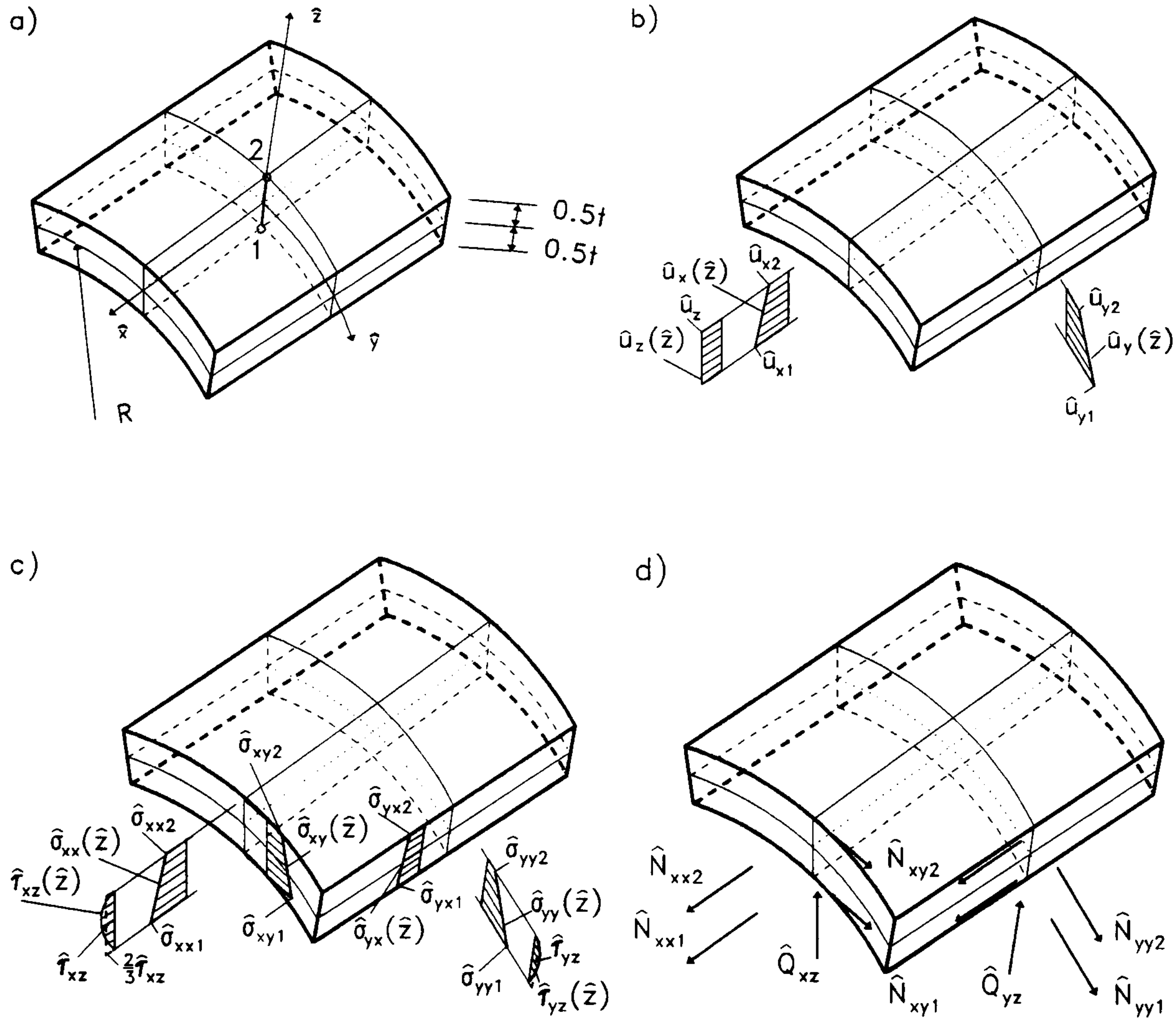

Fig. 3. Illustration of semi-analytical finite element for cylindrical shell:
a) total view; b) distribution of displacements; c) distribution of stresses; d) generalised stresses

The displacement approximation matrix $f(\hat{z}) \equiv f(\xi)$ in (4) is expressed in local co-ordinate ξ by the first order Lagrangian interpolation polynomials $L_i^j(\xi)$

$$f(\xi) = \begin{bmatrix} L_1^2(\xi) & L_2^2(\xi) & 0 & 0 & 0 \\ 0 & 0 & L_1^2(\xi) & L_2^2(\xi) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The linear distribution of the surface stress components $\hat{\sigma}_{xx}(\hat{z})$, $\hat{\sigma}_{yy}(\hat{z})$, $\hat{\tau}_{xy}(\hat{z})$ and parabolic distribution of the transversal stress components $\hat{\tau}_{xz}(\hat{z})$, $\hat{\tau}_{xz}(\hat{z})$ (Fig. 3c) is assumed. The vector of the semi-discrete stresses is now

$$\hat{S}(\hat{x},\hat{y}) \equiv \{\hat{\sigma}_{xx1}(\hat{x},\hat{y}),\ \hat{\sigma}_{xx2}(\hat{x},\hat{y}),\ \hat{\sigma}_{yy1}(\hat{x},\hat{y}),\ \hat{\sigma}_{yy2}(\hat{x},\hat{y}),$$
$$\hat{\tau}_{xy1}(\hat{x},\hat{y}),\ \hat{\tau}_{xy1}(\hat{x},\hat{y}),\ \hat{\tau}_{xz}(\hat{x},\hat{y}),\ \hat{\tau}_{yz}(\hat{x},\hat{y})\}^{t}.$$

Because transversal shear stresses are defined by a single parameter the parabolic distribution is adequate to a constant distribution defined by the values $2\hat{\tau}_{xz}(\hat{z})/3$ and $2\hat{\tau}_{xz}(\hat{z})/3$. The stress approximation matrix $\Phi(\hat{z}) \equiv \Phi(\xi)$ in (9) is defined by the first- and the second order Lagrangian interpolation polynomials.

Formal development of strain approximation (5) leads to two kinds of generalised strains: the basic vector of the model $\hat{\Theta}_2(\hat{x},\hat{y}) \equiv \hat{\Theta}_M(\hat{x},\hat{y})$ and redundant strain vector $\hat{\Theta}_1(\hat{x},\hat{y})$, where

$$\hat{\Theta}_1(\hat{x},\hat{y}) \equiv \{\hat{U}_{x1}(\hat{x},\hat{y}),\ \hat{U}_{x2}(\hat{x},\hat{y}),\ \hat{U}_{y1}(\hat{x},\hat{y}),$$
$$\hat{U}_{y2}(\hat{x},\hat{y}),\ \hat{U}_{z}(\hat{x},\hat{y})\}^{t}$$

and

$$\hat{\Theta}_2(\hat{x},\hat{y}) \equiv \{\hat{\Delta}_{xx1}(\hat{x},\hat{y}),\ \hat{\Delta}_{xx2}(\hat{x},\hat{y}),\ \hat{\Delta}_{yy1}(\hat{x},\hat{y}),$$
$$\hat{\Delta}_{yy\grave{e}}(\hat{x},\hat{y}),\ \hat{\Delta}_{xy1}(\hat{x},\hat{y}),\ \hat{\Delta}_{xy2}(\hat{x},\hat{y}),\ \hat{\Theta}_{xz}(\hat{x},\hat{y}),\ \hat{\Theta}_{yz}(\hat{x},\hat{y})\}^{t}.$$

The corresponding strain approximation matrices in (8) $F_1(\hat{z}) \equiv F_1(\xi) = const$ and $F_2(\hat{z}) \equiv F_2(\xi)$ now are after differentiation expressed in the following form

$$F_1(\xi) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{R} \\ 0 & 0 & 0 & 0 & 0 \\ \frac{1}{t} & -\frac{1}{t} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{t} - \frac{1}{R} & -\frac{1}{t} - \frac{1}{R} & 0 \end{bmatrix};$$

$$F_2(\xi) = \begin{bmatrix} L_1^2(\xi) & 0 & 0 & 0 & 0 \\ L_2^2(\xi) & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{R}L_1^2(\xi) & 0 & 0 & 0 \\ 0 & \frac{1}{R}L_2^2(\xi) & 0 & 0 & 0 \\ 0 & 0 & \left(1 + \frac{1}{R}\right)L_1^2(\xi) & 0 & 0 \\ 0 & 0 & \left(1 + \frac{1}{R}\right)L_2^2(\xi) & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{R} \end{bmatrix}^t$$

By inserting the above matrices F into expression (14) we obtain stress transformation matrix $C(\hat{x}, \hat{y}) \equiv [C_1(\hat{x}, \hat{y}), C_2(\hat{x}, \hat{y})]^t$ from that follows the generalised stress vector $\hat{Q}(\hat{x}, \hat{y})$ contains redundant components. This model leads to the superparametric case.

Here submatrix $C_2(\hat{x}, \hat{y})$ is non-singular and has been taken as modified stress transformation matrix $C_M(\hat{x}, \hat{y}) \equiv C_2(\hat{x}, \hat{y})$. After appropriate matrix manipulations and integration through the thickness it is obtained in quasi-diagonal form

$$C_M(\hat{x}, \hat{y}) = \mathrm{diag}\left[C_{Mxy} \quad C_{Mxy} \quad C_{Mxy} \quad C_{Mz} \quad C_{Mz}\right],$$

where individual submatrices are

$$C_{Mxy}(\hat{x}, \hat{y}) = \frac{1}{t}\begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix} \quad \text{and}$$

$$C_{Mz}(\hat{x}, \hat{y}) = \frac{1}{t}\begin{bmatrix} \frac{3}{2} \end{bmatrix}.$$

Finally, the vector of generalised stresses is defined by the following stress resultants (Fig. 3d) selected as $\hat{Q}_M(\hat{x}, \hat{y}) \equiv \{\hat{N}_{xx1}(\hat{x}, \hat{y}), \hat{N}_{xx2}(\hat{x}, \hat{y}), \hat{N}_{yy1}(\hat{x}, \hat{y}), \hat{N}_{yy2}(\hat{x}, \hat{y}), \hat{N}_{xy1}(\hat{x}, \hat{y}), \hat{N}_{xy2}(\hat{x}, \hat{y}), \hat{Q}_{xz}(\hat{x}, \hat{y}), \hat{Q}_{yz}(\hat{x}, \hat{y})\}^t$ while the set of governing equations (8) for cylindrical shell is represented as follows:

compatibility equations

$$\begin{bmatrix} \frac{\partial}{\partial\hat{x}} & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial\hat{x}} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{R}\frac{\partial}{\partial\hat{y}} & 0 & \frac{1}{R} \\ 0 & 0 & 0 & \frac{1}{R}\frac{\partial}{\partial\hat{y}} & \frac{1}{R} \\ \frac{1}{R}\frac{\partial}{\partial\hat{y}} & 0 & \frac{\partial}{\partial\hat{x}} & 0 & 0 \\ 0 & \frac{1}{R}\frac{\partial}{\partial\hat{y}} & 0 & \frac{\partial}{\partial\hat{x}} & 0 \\ -\frac{1}{t} & \frac{1}{t} & 0 & 0 & \frac{\partial}{\partial\hat{x}} \\ 0 & 0 & -\frac{1}{t} - \frac{1}{R} & \frac{1}{t} - \frac{1}{R} & \frac{1}{R}\frac{\partial}{\partial\hat{y}} \end{bmatrix} \begin{bmatrix} \hat{U}_{x1}(\hat{x}, \hat{y}) \\ \hat{U}_{x2}(\hat{x}, \hat{y}) \\ \hat{U}_{y1}(\hat{x}, \hat{y}) \\ \hat{U}_{y2}(\hat{x}, \hat{y}) \\ \hat{U}_z(\hat{x}, \hat{y}) \end{bmatrix} - \begin{bmatrix} \hat{\Delta}_{xx1}(\hat{x}, \hat{y}) \\ \hat{\Delta}_{xx2}(\hat{x}, \hat{y}) \\ \hat{\Delta}_{yy1}(\hat{x}, \hat{y}) \\ \hat{\Delta}_{yy2}(\hat{x}, \hat{y}) \\ \hat{\Delta}_{xy1}(\hat{x}, \hat{y}) \\ \hat{\Delta}_{xy2}(\hat{x}, \hat{y}) \\ \hat{\Theta}_{xz}(\hat{x}, \hat{y}) \\ \hat{\Theta}_{yz}(\hat{x}, \hat{y}) \end{bmatrix} = O; \qquad (21a)$$

equilibrium equations

$$\begin{bmatrix} \frac{\partial}{\partial \hat{x}} & 0 & 0 & 0 & \frac{1}{R}\frac{\partial}{\partial \hat{y}} & 0 & \frac{1}{t} & 0 \\ 0 & \frac{\partial}{\partial \hat{x}} & 0 & 0 & 0 & \frac{1}{R}\frac{\partial}{\partial \hat{y}} & -\frac{1}{t} & 0 \\ 0 & 0 & \frac{1}{R}\frac{\partial}{\partial \hat{y}} & 0 & \frac{\partial}{\partial \hat{x}} & 0 & 0 & \frac{1}{t}+\frac{1}{R} \\ 0 & 0 & 0 & \frac{1}{R}\frac{\partial}{\partial \hat{y}} & 0 & \frac{\partial}{\partial \hat{x}} & 0 & -\frac{1}{t}+\frac{1}{R} \\ 0 & 0 & -\frac{1}{R} & -\frac{1}{R} & 0 & 0 & \frac{\partial}{\partial \hat{x}} & \frac{1}{R}\frac{\partial}{\partial \hat{y}} \end{bmatrix} \begin{bmatrix} \hat{N}_{xx1}(\hat{x},\hat{y}) \\ \hat{N}_{xx2}(\hat{x},\hat{y}) \\ \hat{N}_{yy1}(\hat{x},\hat{y}) \\ \hat{N}_{yy2}(\hat{x},\hat{y}) \\ \hat{N}_{xy1}(\hat{x},\hat{y}) \\ \hat{N}_{xy2}(\hat{x},\hat{y}) \\ \hat{Q}_{xz}(\hat{x},\hat{y}) \\ \hat{Q}_{yz}(\hat{x},\hat{y}) \end{bmatrix} = - \begin{bmatrix} \hat{p}_{x1}(\hat{x},\hat{y}) \\ \hat{p}_{x2}(\hat{x},\hat{y}) \\ \hat{p}_{y1}(\hat{x},\hat{y}) \\ \hat{p}_{y2}(\hat{x},\hat{y}) \\ \hat{p}_{z}(\hat{x},\hat{y}) \end{bmatrix}; \qquad (21b)$$

constitutive equations

$$\frac{1}{6}\frac{1}{Et} \begin{bmatrix} 20 & -10 & -20\nu & 10\nu & 0 & 0 & 0 & 0 \\ -\nu & 20 & 10\nu & -20\nu & 0 & 0 & 0 & 0 \\ & & 20 & -10 & 0 & 0 & 0 & 0 \\ & & & 20 & 0 & 0 & 0 & 0 \\ & & & & 40(1+\nu) & -20(1+\nu) & 0 & 0 \\ & & & & & 40(1+\nu) & 0 & 0 \\ & sym. & & & & & 12(1+\nu) & 0 \\ & & & & & & & 12(1+\nu) \end{bmatrix} \begin{bmatrix} \hat{N}_{xx1}(\hat{x},\hat{y}) \\ \hat{N}_{xx2}(\hat{x},\hat{y}) \\ \hat{N}_{yy1}(\hat{x},\hat{y}) \\ \hat{N}_{yy2}(\hat{x},\hat{y}) \\ \hat{N}_{xy1}(\hat{x},\hat{y}) \\ \hat{N}_{xy2}(\hat{x},\hat{y}) \\ \hat{Q}_{xz}(\hat{x},\hat{y}) \\ \hat{Q}_{yz}(\hat{x},\hat{y}) \end{bmatrix} - \begin{bmatrix} \hat{\Delta}_{xx1}(\hat{x},\hat{y}) \\ \hat{\Delta}_{xx2}(\hat{x},\hat{y}) \\ \hat{\Delta}_{yy1}(\hat{x},\hat{y}) \\ \hat{\Delta}_{yy2}(\hat{x},\hat{y}) \\ \hat{\Delta}_{xy1}(\hat{x},\hat{y}) \\ \hat{\Delta}_{xy2}(\hat{x},\hat{y}) \\ \hat{\Theta}_{xz}(\hat{x},\hat{y}) \\ \hat{\Theta}_{yz}(\hat{x},\hat{y}) \end{bmatrix} = \boldsymbol{O}. \qquad (21c)$$

Any pair of the semi-discrete generalised stresses related to the nodal points 1 and 2 (in fact, to generatrix surfaces S_1 and S_2) may be changed by resulting force and resulting moment. This observation points to the fact that semi-discrete equilibrium equation (21b) may be expressed in terms of classical membrane forces as well as moments by using simple linear transformation. The same is valid for the other equations (21).

5. Concluding Remarks

An alternative approach - the semi-analytical finite element (SFEM) is proposed for the development of the theory of shells. The method is based on finite element approximation of two independent three-dimensional fields - displacement and stresses in the thickness direction.

The SFEM is applied to develop a set of governing equations of shell. These include compatibility, equilibrium and elasticity relations as well as static and kinematic boundary conditions. A formal technique is put forward to define generalised semi-discrete variables and mixed algebraic-differential operators. The higher order terms may be simply taken into account by refining finite element mesh or by increasing the order of approximation polynomials.

The validity of the method is demonstrated by the example of cylindrical shell.

The SFEM proposed has some principal advantages over classical shell theories:

1) The semi-discrete displacements of shell are compatible with the displacements of three-dimensional body;
2) The governing equations contain only first order derivatives;
3) The method provides extended possibilities for introducing a wider type of loading and supports;
4) The method is independent of geometry of the structure and different types of curved or thick shells may be considered in the same standard manner;
5) The symbolic manipulations and computer algebra may be applied for the derivation of standard operators.

It is clear, however, that comprehensive analysis and future research are needed to develop and extend the SFEM to a wider class of shell problems.

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PUSIAUANALIZINIAI BAIGTINIAI ELEMENTAI IR JŲ NAUDOJIMAS CILINDRINIAMS KEVALAMS MODELIUOTI

R. Kačianauskas

S a n t r a u k a

Pateikiamas pusiauanalizinis baigtinių elementų metodas (PABEM), skirtas netradiciškoms kevalų lygtims išvesti. Metodas numato dviejų nepriklausomų trimačių kintamųjų - poslinkių ir įtempimų laukų aproksimaciją storio kryptimi. Siūlomas metodas yra formalus instrumentas, skirtas pusiaudiskretiniams būvio kintamiesiems konstruoti bei mišriems algebriniams - diferencialiniams lygčių operatoriams sudaryti nepriklausomai nuo kevalo geometrijos bei aproksimavimo laipsnio. Pateikiama technika iliustruojama cilindrinio kevalo pavyzdžiu.

Išvestos alternatyvios tiesinės kevalų teorijos statikos, geometrinių ir lygčių bendrosios išraiškos. Cilindrinio kevalo lygtys pateikiamos kaip atskiras atvejis, gautas įrašius į bendrąsias išraiškas konkretų metrinį tenzorių.

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