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METHOD OF SOLVING NONLINEAR EQUATION SYSTEMS WITH BOOLEAN VARIABLES

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Abstract. A method of solving nonlinear equation systems with Boolean variables, which realizes the strategy of variant-directed enumeration, is related. Necessary and sufficient conditions of feasible plans existence are formalized. A procedure for the formal analysis of subsets of the variants is described. The structure of the algorithm that possesses the completeness property is given. Special cases of systems of equations are considered.

Keywords: Boolean variables, nonlinear equation systems, solution algorithm, directed enumeration.

1. Introduction

The necessity of research of the nonlinear equations systems with Boolean variables (the combinatory equations) appears during derivation of solutions in expert control systems of complex organizational and technological processes and in the diagnostic systems of complex technical objects, in which there can be plural refusals with the effect of "imposing" their consequences (Литвиненко 1992, Литвиненко 2002).

Besides, equation systems with Boolean variables are part of mathematical models of numerous applied combinatory optimization problems, to which belong the problems of routing traffic scheduling, ordering and scheduling interconnected jobs designing complex objects, etc. (Hillier *et al*. 2005, Taha 2006, Winston 2003, Литвиненко 1983, Пападимитриу *et al*. 1985).

Various heuristic algorithms are traditionally used for the solution of similar problems. It is known that such algorithms have a number of grave disadvantages that restrict their practical application. First of all, these include weak action purposefulness and completeness *the absence of the property of completeness* As a result, there are unfairly high machine time expenses and also situations in which there is no success in resolving a problem though the solution obviously exists.

The aspiration to provide the solution process of nonlinear equations systems with Boolean variables on a strict mathematical basis has lead to the development of this new method, which realizes the strategy of the directed enumeration of variants. The statement of this method is the purpose of this article.

2. Problem statement

Nonlinear equation systems with Boolean variables can be represented in the following general view:

$$g_j(x) = b_j;\ j = \overline{1,n}, \qquad (1)$$

where x is the m-dimensional vector of the independent Boolean variables:

$$x = (x_i;\ i = \overline{1,m});\ x_i \in \{0,1\};$$

where $g_j(x)$ is the function of independent variables that has a nonlinear structure:

$$g_j(x) = \sum_{r\in R_j} a_{jr}\varphi_r(x);\ j=\overline{1,n};$$

$\varphi_r(x)$ is a product of independent variables (x-product):

$\varphi_r(x) = \prod_{i\in I_r} x_i;\ r=\overline{1,q};$

R_j is a set of numbers of x-products that are included in the j-th equation; $j=\overline{1,n};$

I_r is a set of numbers of the independent variables that form the r-th x-product; $r=\overline{1,q};$

a_{jr}, b_j are the rational numbers that are not periodic fractions; $j=\overline{1,n};\ r\in R_j$.

It is necessary to define a vector of the bivalent variables' values as $x=(x_i;\ i=\overline{1,m})$ to satisfy the equation system (1).

The problem has a combinatorial nature and belongs to the class of the NP-full problems, which require huge (and, in some practical cases, unacceptable high) expenses of machine time for their solution.

The suggested method allows one to minimize the number of steps involved in the realization of the algorithm and, hence, duration of the solution to the equation system due to high direction and maximal narrowing of the search area of the vector of the values of the Boolean variables $(x_i;\ i=\overline{1,m})$ that satisfy the given system.

The method is based on consecutive breaking of the initial set of variants until the optimal plan or the incompatibility of system of restrictions is established. Allocated subsets of variants are subjected to formal analysis, the purpose of which is:

– to reveal and exclude from further consideration the subsets that do not contain feasible plans;

– to reveal and exclude from further consideration the equations that have lost the property of activity with respect to the plans of the analyzed subset of variants during the solution of the problem;

– to reveal and fix the variables that can accept only non-alternative values (only 0 or only 1) to provide the permissibility of supplemental plans of the analyzed subset of variants.

3. Basic notions and conventional signs

Let us say that at the beginning of a certain step of the problem's (1) solution subtracted λ have not crossed subsets G_k, containing feasible plans, are allocated in the full set of variants G; $k=\overline{1,\lambda}$.

Let I_k^0 and I_k^1 be sets of numbers of the decision variables, which receive the values 0 and 1 in the plans of k-th subset of variants, and let I_k be a set of numbers of variables, the values of which are not fixed in G_k.

The set of variable values $x_i,\ i\in I_k^0\cup I_k^1$ such as $(\forall i\in I_k^0)(x_i=0)$ and $(\forall i\in I_k^1)(x_i=1)$ is called the partial plan of the k-th subset of variants. Any set of variable values $x_i,\ i\in I_k$ satisfying the bivalent condition $x_i\in\{0,1\}$ is called the supplemental plan of subset G_k.

Let us introduce the following signs:

R_k^0 and R_k^1 – the sets of numbers of functions $\varphi_r(x)$ converted by the partial plan of the k-th subset of variants accordingly to 0 and 1:

$$R_k^0=\{r: I_r\cap I_k^0\neq\varnothing\};\ R_k^1=\{r: I_r\subseteq I_k^1\};$$

R_k – the set of numbers of functions $\varphi_r(x)$, that are not converted by the partial plan of subset G_k to a constant:

$$R_k=\{1,\dots,q\}\setminus(R_k^0\cup R_k^1).$$

The constitution of sets R_{jk} and R_{jk}^1, $j=\overline{1,n}$, similar to the mentioned above, but relating to the j-th equation of system (1), is defined according to the formulas:

$$R_{jk}=R_j\cap R_k;\quad R_{jk}^1=R_j\cap R_k^1.$$

The equation of system (1), when even one of the supplemental plans of subset G_k does not satisfy it, is called active with respect to the plans of a given subset. A set of numbers with such restrictions we shall designate as J_k; $J_k\subseteq\{1,2,\dots,n\}$.

Equations system (1), brought into accord with the k-th subset of variants, will have the following form:

$$g_{jk}(x)=b_{jk};\ j\in J_k; \qquad (2)$$

where

$$g_{jk}(x)=\sum_{r\in R_{jk}} a_{jr}\varphi_{rk}(x);$$

$$\varphi_{rk}(x)=\prod_{i\in I_{rk}} x_i;\ I_{rk}=I_r\cap I_k;\ r\in R_k;$$

$$x_i\in\{0,1\};\quad i\in I_k;$$

$$b_{jk}=b_j-\sum_{r\in R_{jk}^1} a_{jr}.$$

On sets R_{jk} and I_{jk}, $j\in J_k$, the following subsets are defined:

$$R_{jk}^2=\{r\in R_{jk}: a_{jr}<0\};$$

$$R_{jk}^2(r')=\{r'\}\cup\{r\in R_{jk}^2: a_{jr}\le a_{jr'}\};$$

$$R_{jk}^3=\{r\in R_{jk}: a_{jr}>0\};$$

$$R_{jk}^3(r'')=\{r''\}\cup\{r\in R_{jk}^3: a_{jr}\ge a_{jr''}\}$$

$$R_{jk}^4(r')=\{r\in R_{jk}^3: I_{rk}\subseteq I_{jk}^2(r')\};$$

$$R_{jk}^5(r'')=\{r\in R_{jk}^2: I_{rk}\subseteq I_{jk}^3(r'')\};$$

$$R_{jk}^6(r'')=\{r\in R_{jk}^2: I_{rk}\cap I_{jk}^8(r'')\neq\varnothing\};$$

$$R_{jk}^7(r')=\{r\in R_{jk}^3: I_{rk}\cap I_{jk}^9(r')\neq\varnothing\};$$

$$R_{jk}^8(r'')=\{r\in R_{jk}^3(r''): m_{rk}=1\};$$

$$R_{jk}^9(r')=\{r\in R_{jk}^2(r'): m_{rk}=1\};$$

$$I_{jk}^v(r')=\bigcup_{r\in R_{jk}^v(r')} I_{rk};\quad v\in\{2,9\};$$

$$I_{jk}^v(r'')=\bigcup_{r\in R_{jk}^v(r'')} I_{rk};\quad v\in\{3,8\},$$

where $m_{rk}=|I_{rk}|$.

Let us introduce the following signs:

$$s_{jk}^{\nu}=\begin{cases}\sum\limits_{r\in R_{jk}^{\nu}} a_{jr}, \text{ if } R_{jk}^{\nu}\neq\varnothing\\ 0, \text{ if } R_{jk}^{\nu}=\varnothing\end{cases};\ \nu\in\{2,3\};$$

$$s_{jk}^{\nu}(r^{*})=\begin{cases}\sum\limits_{r\in R_{jk}^{\nu}} a_{jr}, \text{ if } R_{jk}^{\nu}(r^{*})\neq\varnothing\\ 0, \text{ if } R_{jk}^{\nu}(r^{*})=\varnothing\end{cases},$$

where $r^{*}=\begin{cases}r' \text{ at } \nu\in\{4,7\}\\ r'' \text{ at } \nu\in\{5,6\}\end{cases}$.

For each j-th ($j\in J_k$) equation of the system (2), the necessary and sufficient conditions of its performance can be formulated. The necessary condition is the belonging of the free member b_{jk} to the numerical axis segment restricted to minimal and maximal values of the left part of the equation:

$$\min\ g_{jk}(x)\le b_{jk}\le \max\ g_{jk}(x).$$

With the certain approximation degree as $\min g_{jk}(x)$ and $\max g_{jk}(x)$, the sum of negative and positive coefficients of function $g_{jk}(x)$ accordingly can be accepted:

$$s_{jk}^{2}=\min\ g_{jk}(x)\ ;.\ s_{jk}^{3}=\max\ g_{jk}(x)$$

The sufficient condition of performance of the j-th ($j\in J_k$) equation of system (2) should be the existence of a combination of its coefficients, the sum of which exactly equals b_{jk}. The presence or absence of such a combination we shall reflect as value "true" (1) or "false" (0) of the predicate $P_j(G_k)$ accordingly:

$$P_j(G_k)=\begin{cases}1, \text{ if } \sum\limits_{r\in R_0(G_k)} a_{jr}=b_{jk}\\ 0, \text{otherwise}\end{cases},$$

where $R_j(G_k)$ is some subset of numbers of x-products that are included in the j-th equation of system (2); $R_j(G_k)\subseteq R_{jk}$.

The establishment of the fact of fulfillment or non-fulfillment of a sufficient condition of the existence of a solution to the j-th equation of system is generally equivalent to the solution of the unique linear equation with ρ_{jk} Boolean variables:

$$\sum_{r\in R_{jk}} a_{jr}y_r=b_{jk}, \tag{3}$$

where $y_r\in\{0,1\}$; $r=\overline{1,\rho_{jk}}$; $\rho_{rk}=\left|R_{jk}\right|$.

Because the given equation can be considered a special case of system (1), the simplified modification of the method for its solution is used in this article. Obviously, the presence of a bivalent vector of variable values y_r; $r=\overline{1,\rho_{jk}}$, satisfying equation (3), indicates the fulfillment of a sufficient condition of the existence of a solution to the j-th equation of system and the absence indicates non-fulfilment.

4. Analysis of subsets of variants

The analysis of any subsets of variants G_k ($k=\overline{1,\lambda}$) is based on the establishment of the fact of possibility or impossibility of the observance of necessary and sufficient conditions of the existence of *the feasible decisions of equation set* corresponding to the given subset. In addition, it is necessary to take into account the "side effect"; the non-alternative (with respect to certain equation feasibility) values of variables can be completely unacceptable for other equation of the same system.

Obviously, the specified properties of the solution subsets of the candidates of the problem are defined by equation properties that are included into partial equation systems (2) corresponding to these subsets.

The properties of every k-th subset of variants are formulated as the following statements, the evidence of which relieves the necessity of their proof.

Statement 1. The subset G_k does not contain feasible plans if for some j-th ($j\in J_k$) equation of system (2) one of the following conditions is carried out:

a) $s_{jk}^{2}>b_{jk}$;

b) $s_{jk}^{3}<b_{jk}$;

c) $P_j(G_k)=0$;

d) $(R_{jk}^{2}\neq\varnothing)\,\&\,(\exists r'\in R_{jk}^{2})\cdot\left[(s_{jk}^{2}-a_{jr'}>b_{jk})\,\&\,\left(s_{jk}^{2}+s_{jk}^{4}(r')>b_{jk}\right)\right]$;

e) $(R_{jk}^{3}\neq\varnothing)\,\&\,(\exists r''\in R_{jk}^{3})\cdot\left[(s_{jk}^{3}-a_{jr''}<b_{jk})\,\&\,\left(s_{jk}^{3}+s_{jk}^{5}(r'')<b_{jk}\right)\right]$;

f) $(R_{jk}^{3}\neq\varnothing)\,\&\,(\exists r''\in R_{jk}^{3})\cdot\left[(s_{jk}^{2}+a_{jr''}>b_{jk})\,\&\,\left(s_{jk}^{2}-s_{jk}^{6}(r'')>b_{jk}\right)\right]$;

g). $(R_{jk}^{2}\neq\varnothing)\,\&\,(\exists r'\in R_{jk}^{2})\cdot\left[(s_{jk}^{3}+a_{jr'}<b_{jk})\,\&\,\left(s_{jk}^{3}-s_{jk}^{7}(r')<b_{jk}\right)\right]$

Statement 2. The equation at number j ($j\in J_k$) of system (2) is not active with respect to subset G_k plans if the following condition satisfies it:

$$s_{jk}^{2}=b_{jk}=s_{jk}^{3}=0.$$

Statement 3.1. If some j-th ($j\in J_k$) equation of system (2) is satisfied with the condition:

$$(R_{jk}^{2}\neq\varnothing)\,\&\,(\exists r'\in R_{jk}^{2})\cdot\left[s_{jk}^{2}+s_{jk}^{4}(r')\le b_{jk}<s_{jk}^{2}-a_{jr'}\right],$$

then from subset G_k supplemental plans the feasible plans are only those in which

$$\left[\forall r\in R_{jk}^{2}(r')\right]\left[\varphi_{rk}(x)=1\ \right].$$

Statement 3.2. If some j-th ($j\in J_k$) equation of system (2) is satisfied with the condition:

$$(R_{jk}^{3}\neq\varnothing)\,\&\,(\exists r''\in R_{jk}^{3})\cdot\left[s_{jk}^{3}-a_{jr''}<b_{jk}\le s_{jk}^{3}+s_{jk}^{5}(r'')\right],$$

then from subset G_k supplemental plans the feasible plans are only those in which

$$\left[\forall r\in R_{jk}^{3}(r'')\right]\left[\varphi_{rk}(x)=1\ \right].$$

Statement 4.1. If some *j*-th ($j \in J_k$) equation of system (2) is satisfied with the condition:

$$(R_{jk}^3 \neq \varnothing) \& (\exists r'' \in R_{jk}^3) \cdot \left[s_{jk}^2 - s_{jk}^6(r'') \leq b_{jk} < s_{jk}^2 + a_{jr''} \right],$$

then from subset G_k supplemental plans the feasible plans are only those in which

$$\left[\forall r \in R_{jk}^3(r'') \right] \left[\varphi_{rk}(x) = 0 \right].$$

Statement 4.2. If some *j*-th ($j \in J_k$) equation of system (2) is satisfied with the condition:

$$(R_{jk}^2 \neq \varnothing) \& (\exists r' \in R_{jk}^2) \cdot \left[s_{jk}^3 + a_{jr'} < b_{jk} \leq s_{jk}^3 - s_{jk}^7(r') \right],$$

then from subset G_k supplemental plans the feasible plans are only those in which

$$\left[\forall r \in R_{jk}^2(r') \right] \left[\varphi_{rk}(x) = 0 \right].$$

The procedure of analysing the k-th subset of the problem's candidate solutions is based on a consecutive check on the fulfilment of each statement condition for all equations of system (2). Depending on the results, in the analysis cycle this or that sequence of actions is carried out.

1. If the condition of statement 1 is performed for some equation of system (2), the analyzed subset of variants G_k is excluded from further consideration as not containing feasible plans and the analysis procedure is completed. Otherwise, the following item of the given procedure is carried out.

2. If the j^* th ($j^* \in J_k$) equation of system (2) is satisfied by the condition of statement 2, it is excluded from the given system, as it cannot influence the choice of the supplemental plan of the *k*-th subset of variants. After that, the set J_k, elements of which identify the equations active with respect to plans G_k, is corrected. The updated set is defined according to the formula:

$$J_k^* = J_k \setminus \{j^*\}.$$

Furthermore, if the equation that is considered is not the last in system (2), the condition of statement 2 is checked for the next equation, etc.

If $J_k^* = \varnothing$, that means that all supplemental plans of the subset of variants G_k satisfy the system of equations (2). This ends the computing process because the solution of the initial system of equations (1) is found. The solution is the decision variables value vector, consisting of partial and any supplemental plans of subset G_k.

If after checking the fulfilment of the condition of statement 2 for all equations of system (2), it appears that $J_k^* \neq \varnothing$, then the transition to the next step of the procedure of analysing the subset of variants G_k is made.

3. If the *j*-th $(j \in J_k^*)$ equation of system (2) and some $r' \in R_{jk}^2$ are satisfied with the condition of statement 3.1, the variables x_i, $i \in I_{jk}^2(r')$ are set to 1s. These values are substituted in all active (with respect to the plans of subset G_k) equations of system (2). After that the repeated analysis cycle of the *k*-th subset of variants is made starting with the first item. The performance check of the condition of statement 1 for the given inequality in a repeated cycle is omitted.

Otherwise, the next step in the analysis of subset G_k is carried out.

4. If the j-th $(j \in J_k^*)$ equation of system (2) and some $r'' \in R_{jk}^3$ are satisfied with the condition of statement 3.2, then the variables x_i, $i \in I_{jk}^3(r'')$ are set to 1s. These values are substituted in all active (with respect to subset G_k plans) equations of system (2). After that the repeated analysis cycle of the *k*-th subset of variants is made starting with the first item. The performance check of the condition of statement 1 for the given inequality in a repeated cycle is omitted.

Otherwise, the next step in the analysis of subset G_k is carried out.

5. If the *j*-th $(j \in J_k^*)$ equation of system (2) and some $r'' \in R_{jk}^3$ are satisfied with the condition of statement 4.1, then the variables x_i, $i \in I_{jk}^8(r'')$ are set to 0s. These values are substituted in all active (with respect to subset G_k plans) equations of system (2). After that the repeated analysis cycle of the *k*-th subset of variants is made starting with the first item. The performance check of the condition of statement 1 for the given inequality in a repeated cycle is omitted.

Otherwise, the next step in the analysis of subset G_k is carried out.

6. If the *j*-th $(j \in J_k^*)$ equation of system (2) and some $r' \in R_{jk}^2$ are satisfied with the condition of statement 4.2, then the variables x_i, $i \in I_{jk}^9(r')$ are set to 0s. These values are substituted in all active (with respect to subset G_k plans) equations of system (2). After that the repeated analysis cycle of the *k*-th subset of variants is made starting with the first item. The performance check of the condition of statement 1 for the given inequality in a repeated cycle is omitted.

To check the performance of the conditions of the statements 3.1 and 4.2 for the next j-th equation, starting by considering the minimal (negative) coefficient of function $g_{jk}(x)$ as $a_{jr'}$ is recommended. Subsequently, if at this $a_{jr'}$ the condition of the given statement is fulfilled, it is expedient to use the maximal negative coefficient of function $g_{jk}(x)$, for which the condition $s_{jk}^2 - a_{jr'} > b_{jk}$ satisfies, as this parameter.

The fulfilment of the conditions of statements 3.2 and 4.1 for the next j-th equation is originally checked for the case when the maximal (positive) coefficient of function $g_{jk}(x)$ is taken as $a_{jr''}$. If at this $a_{jr''}$, the condition of the given statement is fulfilled, then the minimal positive coefficient of function $g_{jk}(x)$ satisfying the condition $s_{jk}^2 + a_{jr''} > b_{jk}$ is considered this parameter.

The choice of parameters $a_{jr'}$ and $a_{jr''}$ allows cutting off from G_k the greatest capacity subsets of variants that do not contain feasible plans.

The procedure of analysing the variant subset G_k stops in the following cases:

a) if it appears that the subset G_k does not contain feasible plans;

b) if the set of equation numbers active with respect to plans of *k*-th subset of variants becomes empty;

c) if in the last analysis cycle, none of the variables $x_i, i \in I_k$ are given fixed values.

5. The structure of the algorithm

The algorithm that solves equation system (1), that realizes the strategy of the directed enumeration of variants, provides the performance at each computing process stage the following sequence of actions:

– choosing a subset of variants subject to further splitting;

– choosing a variable, the values of which are subject to fixing;

– splitting a subset of variants into two not crossing subsets;

– analysing new received subsets of variants;

– checking the conditions of end of the computing process.

1. Choosing a subset of variants subject to splitting.

Since the problem that is being considered is not an optimization one, it is expedient to use the number of variables that have fixed values in subsets G_k, $k = \overline{1, \lambda}$ as criteria for choosing a subset of variants for further splitting. This means that for further splitting the subset of variants G_{k^*}, $1 \le k^* \le \lambda$ is chosen, to which the partial plan of the maximal capacity corresponds:

$$\mu_{k^*} = \max\left\{\mu_k, k = \overline{1, \lambda}\right\},$$

where $\mu_k = \left|I_k^0 \cup I_k^1\right|$.

This criterion responds to the aspiration to achieve the result of required calculations for a minimum quantity of algorithmic steps.

2. Choosing a variable, the values of which are subject to fixing.

According to the reasons given in the previous step, it is expedient for the given operation to choose a variable, the fixing of values of which can lead to essential simplification of system (2) corresponding to the subset of variants G_{k^*}. Any variable of that *x*-product that belongs to equation set (2) with the maximal coefficient in absolute value can have such a property.

Hence, for giving values 0 and 1 an arbitrary variable, $x_{i^*} \in I_{r^*k^*}$, which belongs to the *x*-product $\varphi_{r^*k^*}(x)$, is chosen, so that

$$\left|a_{jr^*}\right| = \max\{\left|a_{jr}\right|;\; j \in J_{k^*}; r \in R_{jk^*}\}.$$

3. Splitting a subset of variants G_{k^*}.

By fixing the values of variable x_{i^*}, the subset G_{k^*} is broken into two not crossing subsets of variants $G_{k^*}^0$ and $G_{k^*}^1$. In all plans of the first of them, $x_{i^*} = 0$, and in all plans of the second $x_{i^*} = 1$. These values are serially substituted in equations of system (2), with the help of which two new systems of equations corresponding to the two new subsets of variants $G_{k^*}^0$ and $G_{k^*}^1$ are formed.

4. Analysing subsets of variants $G_{k^*}^0$ and $G_{k^*}^1$.

The new subsets $G_{k^*}^0$ and $G_{k^*}^1$ are serially exposed to the formal analysis according to the procedure stated above. After that (if the required solution is not found), all considerations left in the subset of variants are renumbered again, starting with one.

5. Checking the conditions of the end of the computing process.

The computing process ends after finding a solution (a set of solutions) to equation set (1) or after establishing its incompatibility.

Preassigned equation set has a unique solution if after the next searching stage only one subset of variants with the unique equation of system (2), which satisfies the condition of one of the statements 3.1-4.2, is left:

$$(\lambda = k = 1) \,\&\, (\left|J_k\right| = 1) \,\&\, \{(\left|R_{jk}\right| = 1) \,\&\, (a_{jr} = b_{jk}) \vee [(R_{jk}^2 = \varnothing) \,\&\, (R_{jk}^3 \ne \varnothing) \vee (R_{jk}^2 \ne \varnothing) \,\&\, (R_{jk}^3 = \varnothing)] \,\&\, (\forall r \in R_{jk})(m_{rk} = 1) \,\&\, (b_{jk} = 0)\}.$$

A formal attribute of the existence of more than one solution of system (1) is the absence of equations in system (2) that are active with respect to the plans of some subset of variants involving variables that have not yet received fixed values:

$$(\lambda \ge 1) \,\&\, (\exists k : 1 \le k \le \lambda)[(J_k = \varnothing) \,\&\, (I_k \ne \varnothing)].$$

In particular, if for some subset G_k; $1 \le k \le \lambda$ the condition $(J_k = \varnothing) \,\&\, (I_k \ne \varnothing)$ is satisfied, that means that the given subset contains as many solutions of equations set (1) as supplemental plans.

A formal attribute of the incompatibility of equation set (1) is the absence of subsets of variants left after performing the analysis at any step of the computing process: $\lambda = 0$.

If at the current step the conditions of ending the computing process are not satisfied, the following step of the described algorithm is carried out.

It is expedient to begin the problem solving with an analysis of the full set of variants G. In some cases, this allows one to determine the solution of equation set (1) without the splitting procedure, to establish the fact of its incompatibility, or at least to narrow the area of the search for a solution.

6. Special cases

In many management and diagnosis problems, models with a unimodular coefficient matrix and the whole non-negative right parts of equation set (1) can frequently be found:

$(\forall j : 1 \le j \le n)[(\forall r \in R_j)(a_{jr} = 1) \,\&\, (b_j \ge 0)]$.

In this case, the necessary and sufficient conditions for the fulfilment of the j-th ($j \in J_k$) equation of set (2) are expressed by the formula

$$0 \le b_{jk} \le s_{jk}, \tag{4}$$

where $s_{jk} = \begin{cases} |R_{jk}|, & \text{if } R_{jk} \ne \varnothing \\ 0 & \text{otherwise} \end{cases}$,

and the above statements, postulating properties of the k-th $(k = \overline{1, \lambda})$ subset of variants, get the following formulation.

Statement 1'. Subset G_k does not contain feasible plans if some j-th ($j \in J_k$) equation of set (2) is satisfied with this condition:

$$(b_{jk} < 0) \vee (b_{jk} > s_{jk}).$$

Statement 2'. The equation at number j ($j \in J_k$) of set (2) is not active with respect to subset G_k plans if it is satisfied with this condition:

$$b_{jk} = s_{jk} = 0.$$

Statement 3'. If some j-th ($j \in J_k$) equation of system (2) is satisfied with this condition:

$$s_{jk} = b_{jk} > 0,$$

then from the supplemental plans of subset G_k only those in which $(\forall r \in R_{jk})\,[\varphi_{rk}(x) = 1]$ can be feasible.

Statement 4'. If some j-th ($j \in J_k$) equation of system (2) is satisfied with this condition:

$$b_{jk} = 0,$$

then from the supplemental plans of subset G_k only those in which $(\forall r \in R_{jk})\,[\varphi_{rk}(x) = 0]$ can be feasible.

Let

$$I_{jk} = \bigcup_{r \in R_{jk}} I_{rk}\,;\; R'_{jk} = \{r \in R_{jk} :\; m_{rk} = 1\}\,;$$

$$I'_{jk} = \bigcup_{r \in R'_{jk}} I_{rk}\,.$$

The general structure of the algorithm used to solve equation set (1) in the case of unimodular coefficient matrix remains the former, but the procedure of analysing subsets of variants is reduced up to four steps.

The basic action of the third step (replacing the third and fourth steps of the general procedure) in the special case considered is setting the value of 1 to variables x_i, $i \in I_{jk}$ if the j-th $(j \in J_k^*)$ equation of system (2) is satisfied with the condition of statement 3'.

In the fourth step (replacing the fifth and sixth steps of the general procedure), one must set the value of 0 to variables x_i, $i \in I'_{jk}$, if the j-th $(j \in J_k^*)$ equation of system (2) is satisfied with the condition of statement 4'.

The other special case that also frequently occurs in diagnosing problems of complex objects with plural refusals is characterized by the unimodality of not only the *matrix of the coefficients*, but also the vector free member:

$(\forall j : 1 \le j \le n)[(\forall r \in R_j)(a_{jr} = 1) \,\&\, (b_j \in \{0,1\})]$.

In this case, the necessary and sufficient conditions for the fulfilment of the j-th ($j \in J_k$) equation of system (2) are expressed, as earlier, by formula (4), and the conditions of four statements postulating the properties of the k-th $(k = \overline{1, \lambda})$ subset of variants are formulated as follows:

– for statement 1': $(b_{jk} < 0) \vee (s_{jk} = 0) \,\&\, (b_{jk} = 1)$;

– for statement 2': $s_{jk} = b_{jk} = 0$;

– for statement 3': $s_{jk} = b_{jk} = 1$;

– for statement 4': $(s_{jk} > 0) \,\&\, (b_{jk} = 0)$.

The fulfilment of the condition of statement 3' proves that the left part of the j-th equation of system (2) will consist of the unique x-product $\varphi_{rk}(x)$. In this case, all variables x_i, $i \in I_{rk}$ forming the given x-product are set to unique feasible values of 1s, and the equation turns to identity and is excluded from further consideration.

The fulfilment of the condition of statement 4' requires transformation the left part of the j-th equation of system (2). In this case, all variables x_i, $i \in I'_{jk}$ are set values of 0s.

7. Conclusion

Problems that require solving nonlinear equation systems with Boolean variables are widely used in automated control systems and the design and diagnostics of the systems of complex objects.

In order to solve such problems, various heuristic algorithms are traditionally used, but they have limited practical application due to their known disadvantages.

In this article, a mathematical method to solve nonlinear equation systems with Boolean variables is set forth. This method realizes the strategy of the directed enumeration of variants.

The given algorithm has the property of completeness because none of the allocated subsets of variants are excluded from consideration until the incompatibility of the corresponding set of equations is established.

The computer realization of the method was carried out in the UNIX IP operational environment with the use of the language C ++.

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NETIESINIŲ LYGČIŲ SISTEMŲ SPRENDIMAS SU BŪLIO KINTAMAISIAIS

O. Lytvynenko

S a n t r a u k a

Netiesinių lygčių sistemos su Būlio kintamaisiais sprendžiamos naudojant variantų kryptinės numeracijos metodą. Formalizuojamos būtinos ir pakankamos galimų schemų sąlygos. Aprašomos variantų poeibių formalios analizės procedūros. Pateikiama struktūra algoritmo, turinčio užbaigtumo savybių. Svarstomos lygčių sistemų atskiri atvejai.

Reikšminiai žodžiai: Būlio kintamieji, netiesinių lygčių sistemos, sprendimo algoritmas, kryptinė numeracija.